4. Extensions and Generalizations

In the preceeding section on the basis of a Hilbert scale we had derived relations between the approximation quantities $\varkappa_{\alpha\beta}$ and inverse quantities $\sigma_{\alpha\beta}$ for different α, β and moreover the simultaneous approximability within a fixed range of the scale. The question arises what properties of a sequence $\{X_\alpha \mid \alpha \in A\}$ of Banach spaces are necessary such that these assertions are valid.

In Lemma 3.3 and Corollary 3.4 a certain constant enters the right hand sides of the estimates. According to figure 2 we have:

Let $\underline{c} < c < \bar{c}$ be fixed and assume $\varkappa$ to be normalized such that for $x \in H_{\bar{c}}$

$$\inf_{\xi \in S} \|x-\xi\|_c \leq \varkappa^{\bar{c}-c} \|x\|_{\bar{c}} \tag{4.1}$$

holds. Then there is a $\xi \in S$ such that

$$\|x-\xi\|_b \leq C \varkappa^{\bar{c}-b} \|x\|_{\bar{c}} \tag{4.2}$$

holds uniformly for $b \in [\underline{c}, c]$.

The constant is given by

$$C = 2\left\{1 + 2^{\gamma/(1-\gamma)}\right\} \quad \text{with} \quad \gamma = \frac{c-\underline{c}}{\bar{c}-\underline{c}} . \tag{4.3}$$

If we think of $\underline{c} \ll c \approx \bar{c}$ then C will be very large. In this way for large intervals the simultaneous approximability has its prize in bad constants. The second question to be discussed in this section is: Does an approximation $\xi \in S$

exist such that for all $b \le c$ estimates of the type

$$\|x-\xi\|_b \le C(b)\, \varkappa^{c-b} \|x\|_c \tag{4.4}$$

will hold.

We go back to the first question. Let us look at the proofs of the lemmata etc. until 3.10 in Section 3. Actually we have used only the Lemmata 2.2, 2.3, and 2.4, i.e. besides the inclusions $H_\beta \subseteq H_\alpha$ for $\alpha < \beta$ the logarithmic convexity of the norm and an appropriate approximability of elements $x \in H_\beta$ by elements $y \in H_\gamma$ with $\gamma > \beta$.

<u>Remark:</u> In this context the compactness of the embedding $H_\beta \to H_\alpha$ is irrelevant.

Now let us assume that a set $\{X_\alpha \mid \alpha \in A\}$ of Banach spaces with norms $\|\cdot\|_\alpha$ is given which fulfills

<u>Proposition 4.1:</u> For $\alpha, \beta \in A$ with $\alpha < \beta$ the space X_β is continuously embedded in X_α and

$$\|x\|_\alpha \le \|x\|_\beta \quad \text{for } x \in X_\beta . \tag{4.5}$$

<u>Proposition 4.3:</u> For any triple $\alpha, \beta, \gamma \in A$ with $\alpha < \beta < \gamma$ then

$$\|x\|_\beta \le \|x\|_\alpha^\mu \, \|x\|_\gamma^\nu \quad \text{for } x \in H_\gamma \tag{4.6}$$

with

$$\mu = \frac{\gamma-\beta}{\gamma-\alpha}, \quad \nu = \frac{\beta-\alpha}{\gamma-\alpha} . \tag{4.7}$$

<u>Proposition 4.4:</u> Let α, β, γ as stated in Proposition 4.3. To $t > 0$ and $x \in H_\beta$ there is an approximation $y \in H_\gamma$ according to

$$\begin{aligned} \|x-y\|_\alpha &\le t^{\beta-\alpha} \|x\|_\beta , \\ \|x-y\|_\beta, \ \|y\|_\beta &\le \|x\|_\beta , \\ \|y\|_\gamma &\le t^{-(\gamma-\beta)} \|x\|_\beta . \end{aligned} \tag{4.8}$$

It is obvious to verify that then the assertions of the Lemmata and Corollaries 3.3, 3.4, 3.5, 3.8, and 3.9 remain valid, of course the indices α, β etc. have to be in the set A. In the inequalities of the propositions some numerical constants may appear. All the assertions remain valid with modified constants.

<u>Remark:</u> In the applications quite often it is possible to check the validity of the propositions directly. We mention the case of Sobolev spaces $\{W_p^k \mid k = 0,1,\ldots\}$ for fixed $p \in (0,\infty)$.

Now we turn over to the second question. In order to have transparency we will use a rescaling: The linear transformation $b \to \hat{b}$ defined by $\hat{b} = (b-c)|(\bar{c}-c)$ maps the points $b = c$ resp. $b = \bar{c}$ to $\hat{b} = 0$ resp. $\hat{b} = 1$ and $b < c$ to $\hat{b} < 0$. By a corresponding rescaling of $\varkappa$ we replace (4.1) by

<u>Assumption:</u> Let $\varkappa$ be normalized such that for $x \in H_1$

$$\inf_{\xi \in S} \|x-\xi\|_0 \le \varkappa \|x\|_1 \tag{4.9}$$

holds.

According to (4.4) we ask for a $\xi \in S$ such that

$$\|x-\xi\|_b \le C(b)\, \varkappa^{-b} \|x\| \qquad (4.10)$$

holds true for $b \le 0$. The answer is given by

<u>Theorem 4.5:</u> Let $\varkappa$ be defined by (4.9). To $x \in H_0$ there is a $\xi \in S$ such that (4.10) holds with $C(b)$ depending only on b for $b \le 0$.

<u>Proof:</u> We recall the definition of the α-inner-product and α-norm (2.17-18). For $\alpha < 0$ and $|\alpha| >> 1$ the Fourier-coefficients $x_i = (x,\varphi_i)$ contribute to the α-norm with a factor λ_i^{α} . Because of $\lambda_i \to \infty$ for $i \to \infty$ these factors will be arbitrary small. We speak of a polynomial decay. Now we introduce an additional inner product resp. norm by

$$\begin{aligned} (x,y)_{(t)} &= \Sigma\, e^{-\sqrt{\lambda_i}\,t} (x,\varphi_i)(y,\varphi_i) \\ \|x\|_{(t)} &= (x,x)_{(t)}^{1/2} \end{aligned} \qquad (4.11)$$

for $t > 0$. Now the factors $\exp(-\sqrt{\lambda_i}\,t)$ have an exponential decay. Obviously we have

$$\|x\|_{(t)} \le c(\alpha,t)\, \|x\|_{\alpha} \quad \text{for} \quad x \in H_{\alpha} \qquad (4.12)$$

with $c(\alpha,t)$ depending only on α and $t > 0$. Thus the (t)-norm is weaker than any α-norm. On the other hand any negative norm, i.e. $\|\cdot\|_{\alpha}$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (t)-norm:

<u>Lemma 4.6:</u> Let $\alpha > 0$ be fixed. The $(-\alpha)$-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 \qquad (4.13)$$

with $\delta > 0$ being arbitrary.

<u>Proof:</u> For any $t,\delta,\alpha > 0$ and $\lambda \ge 1$ the inequality

$$\lambda^{-\alpha} \le \delta^{2\alpha} + e^{t(\delta^{-1}-\sqrt{\lambda})} \qquad (4.14)$$

holds for the following reason: If $\lambda^{-1/2} \le \delta$ then obviously $\lambda^{-\alpha} \le \delta^{2\alpha}$. In case of $\lambda^{-1/2} \ge \delta$ then we have $\exp\{t(\delta^{-1}-\sqrt{\lambda})\} \ge 1$ whereas $\lambda^{-\alpha} \le 1$ is a consequence of $\alpha > 0$, $\lambda \ge 1$. By the aid of (4.14) we find

$$\begin{aligned} \|x\|_{-\alpha}^2 &= \Sigma\, \lambda_i^{-\alpha}\, x_i^2 \\ &\le \delta^{2\alpha}\, \Sigma\, x_i^2 + e^{t/\delta}\, \Sigma\, e^{-t\sqrt{\lambda_i}}\, x_i^2 \ . \end{aligned} \qquad (4.15)$$

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<u>Remark:</u> (4.13) is in a certain sense the counterpart of the logarithmic convexity of the α-norms: We go back to (2.33) which we rewrite with $\varepsilon > 0$ in the form

$$\begin{aligned} \|x\|_{\beta}^2 &\le (\|x\|_{\alpha}^2)^{\mu}(\|x\|_{\gamma}^2)^{\nu} \\ &\le (\varepsilon^{-\nu/\mu}\|x\|_{\alpha}^2)^{\mu}(\varepsilon\|x\|_{\gamma}^2)^{\nu} \ . \end{aligned} \qquad (4.16)$$

Because of $\mu,\nu > 0$ and $\mu + \nu = 1$ we may introduce $p = \mu^{-1}$, $q = \nu^{-1}$ and apply Young's inequality.

We get

$$\|x\|_\beta^2 \le \nu\cdot\varepsilon \, \|x\|_\gamma^2 + \mu\varepsilon^{-\nu/\mu}\|x\|_\alpha^2 \quad . \tag{4.17}$$

The counterpart of Lemma 2.4 is:

<u>Lemma 4.7:</u> Let $t, \delta > 0$ be fixed. To any $x \in H_o$ there is an $y \in H_1$ according to

$$\begin{aligned} \|x-y\| &\le \|x\| \quad , \\ \|y\|_1 &\le \delta^{-1}\|x\| \quad , \\ \|x-y\|_{(t)} &\le e^{-t/\delta}\|x\| \quad . \end{aligned} \tag{4.18}$$

<u>Proof:</u> We try to use

$$y = \sum_{i=1}^{N} x_i \, \varphi_i \tag{4.19}$$

with $x_i = (x, \varphi_i)$ and N chosen appropriately. Then (4.18_1) is satisfied. Further we get

$$\begin{aligned} \|y\|_1^2 &= \sum_1^N \lambda_i \, x_i^2 \\ &\le \lambda_N \, \|x\|^2 \; . \end{aligned} \tag{4.20}$$

In order that (4.18_2) holds true we may choose N according to

$$\lambda_N \le \delta^{-2} < \lambda_{N+1} \quad . \tag{4.21}$$

Then we get

$$\begin{aligned} \|x-y\|_{(t)}^2 &= \sum_{N+1}^{\infty} e^{-t \, \lambda_i^{1/2}} \, x_i^2 \\ &\le e^{-t \, \lambda_{N+1}^{1/2}} \, \|x\|^2 \quad . \end{aligned} \tag{4.22}$$

Therefore also (4.18_3) is valid.

Now we come to the counterparts of Lemma 3.3:

<u>Lemma 4.8:</u> Let $\varkappa$ be defined by (4.9). Then

$$\begin{aligned} E_t(x) &:= \inf_{\xi \in S} \left\{ e^{-t/2\varkappa}\|x-\xi\| + \|x-\xi\|_{(t)} \right\} \\ &\le 4e^{-t/2\varkappa}\|x\| \quad . \end{aligned} \tag{4.23}$$

<u>Proof:</u> We define for $t > 0$ fixed

$$\varepsilon = \varepsilon_t = \sup\left\{ E_t(x) \mid x \in H \cap \|x\| = 1 \right\} \tag{4.24}$$

what gives (trivially) for $x \in H$

$$E_t(x) \le \varepsilon\|x\| \quad . \tag{4.25}$$

Since $E_t(x) = E_t(x-\eta)$ for $\eta \in S$ we have in case of $x \in H_1$

$$E_t(x) \le \varepsilon \inf_{\eta \in S} \|x-\eta\| \le \varepsilon \, \varkappa\|x\|_1 \quad . \tag{4.26}$$

Further we get because of

$$E_t(x) \le E_t(x-y) + E_t(y) \tag{4.27}$$

with y chosen according to Lemma 4.7

$$\begin{aligned} E_t(x) &\le \|x-y\|_{(t)} + e^{-t/2\varkappa}\|x-y\| + E_t(y) \\ &\le \left\{ e^{-t/\delta} + e^{-t/2\varkappa} \right\} \|x\| + E_t(y) \end{aligned} \tag{4.28}$$

and using (4.26) in combination with (4.18_2)

$$(4.29) \qquad E_t(x) \le \left\{e^{-t/\delta} + e^{-t/2\kappa} + \varepsilon\,\kappa\,\delta^{-1}\right\}\|x\| \ .$$

The choice $\delta = 2\kappa$ gives

$$(4.30) \qquad E_t(x) \le (2e^{-t/2\kappa} + \tfrac{1}{2}\varepsilon)\ \|x\| \ .$$

respective

$$(4.31) \qquad \varepsilon \le 2e^{-t/2\kappa} + \tfrac{1}{2}\varepsilon \ . \qquad \#$$

In this way we have proven the simultaneous approximability: To $x \in H$ there is a $\xi \in S$ according to

$$(4.32) \qquad \begin{aligned} \|x-\xi\| &\le 4\|x\| \ , \\ \|x-\xi\|_{(t)} &\le 4e^{-t/2\kappa}\|x\| \ . \end{aligned}$$

After these preparations we are ready to finish the proof of Theorem 4.5. Lemma 4.6 in conjunction with (4.32) gives

$$(4.33) \qquad \|x-\xi\|_{-\alpha}^2 \le 16\left\{\delta^{2\alpha} + e^{t(\delta^{-1}-1/\kappa)}\right\}\ \|x\|^2$$

for $t, \delta > 0$ arbitrarily chosen. We take $\delta = 2\kappa$ leading to

$$(4.33) \qquad \|x-\xi\|_{-\alpha}^2 \le 16\left\{2^{2\alpha}\,\kappa^{2\alpha} + e^{-t/2\kappa}\right\}\ \|x\|^2 \ .$$

Now we take t large enough such that

$$(4.34) \qquad e^{-t/2\kappa} \le \kappa^{2\alpha}$$

which is possible for any $\kappa > 0$. This leads to (4.10). #