3. Approximation Theory in Hilbert Scales

In the following $\{H_\alpha\}$ denotes a Hilbert scale as discussed in the preceding section. For simplicity we will restrict ourselves to $\alpha \in [0,1] =: I$. Further, let S denote an appropriate 'approximation space'. Similarily for simplicity we assume $S \subset H_1$. Given an x not in S we may ask for

$$(3.1) \qquad d(x,S) = \text{dist}\,(x,S) = \inf_{\xi \in S} \|x-\xi\| \quad .$$

In our case the norm may be any α-norm. Moreover, one asks not only for the approximability of one special element but of a whole class of them. In the applications this is characterized by a condition $x \in H_\beta$ with some index β . Since

$$(3.2) \qquad d(\lambda x,S) = |\lambda|\, d(x,S)$$

the distance is to be compared with the norm of x . Therefore the smallest constant $\varkappa = \varkappa_{\alpha\beta}$ is sought such that for all $x \in H_\beta$

$$(3.3) \qquad \inf_{\xi \in S} \|x-\xi\|_\alpha \le \varkappa_{\alpha\beta} \|x\|_\beta$$

holds. This is given by

$$(3.4) \quad \varkappa_{\alpha\beta} := \sup\left\{\inf_{\xi \in S} \|x-\xi\|_\alpha \,\middle|\, x \in H_\beta \wedge \|x\|_\beta = 1\right\} \quad .$$

Of course, $\varkappa_{\alpha\beta}$ is only defined for $\alpha \le \beta$, otherwise they are $+\infty$. Because of Lemma 2.2 we get

Lemma 3.1: $\varkappa_{\alpha\beta}$ is monotone nondecreasing in the first and nonincreasing in the second argument.

We will show a number of relations for $\varkappa_{\alpha\beta}$ concerning different indizes. The first is

Lemma 3.2: Let $\alpha < \beta < \gamma$. Then

$$\varkappa_{\alpha\gamma} \le \varkappa_{\alpha\beta}\,\varkappa_{\beta\gamma} \quad . \tag{3.5}$$

Proof: Since S is a linear subspace we have for $\eta \in S$ arbitrary

$$\inf_{\xi \in S} \|x-\eta-\xi\|_\alpha = \inf_{\xi \in S} \|x-\xi\|_\alpha \tag{3.6}$$

and therefore

$$\inf_{\xi \in S} \|x-\xi\|_\alpha \le \varkappa_{\alpha\beta} \|x-\eta\|_\beta \quad . \tag{3.7}$$

Now we may choose η such that

$$\|x-\eta\|_\beta = \inf_{\xi \in S} \|x-\xi\|_\beta \le \varkappa_{\beta\gamma} \|x\|_\gamma \quad . \tag{3.8}$$

#

The lemma may be interpreted in the following way: Let α, β, γ be given with $0 \le \alpha < \beta < \gamma \le 1$ and let $x \in H_1$ be fixed. Then there are elements $\xi_\alpha, \xi_\beta, \xi_\gamma \in S$ such that

$$\|x-\xi_\alpha\|_\alpha \le \varkappa_{\alpha\beta} \|x\|_\beta \quad ,$$

$$\|x-\xi_\beta\|_\beta \le \varkappa_{\beta\gamma} \|x\|_\gamma \quad , \tag{3.9}$$

$$\|x-\xi_\gamma\|_\alpha \le \varkappa_{\alpha\beta}\,\varkappa_{\beta\gamma} \|x\|_\gamma \quad .$$

Up to now there is no indication whether or not in the two last inequalities ξ_β equals ξ_γ. Without doubt the question of simultaneous approximability is of special interest. The essential key is

Lemma 3.3: Let γ be fixed with $0 < \gamma < 1$ and define

$$\varkappa = \varkappa_{\gamma 1}^{1/(1-\gamma)} \quad . \tag{3.10}$$

Then

$$\inf_{\xi \in S} \left\{ \|x-\xi\|_0 + \varkappa^\gamma \|x-\xi\|_\gamma \right\} \le c\,\varkappa \|x\|_1 \tag{3.11}$$

with $c = 2\left\{1+2^{\gamma/(1-\gamma)}\right\}$.

Before proving the lemma we remark the following: The introductions of $\varkappa$ has a 'rescaling' reason. We have

$$\inf_{\xi \in S} \|x-\xi\|_\gamma \le \varkappa^{1-\gamma} \|x\|_1 \quad . \tag{3.12}$$

Now the lemma guarantees the existence of an element $\xi \in S$ such that

$$\|x-\xi\|_0 \le c\,\varkappa \|x\|_1 \quad ,$$

(3.13)

$$\|x-\xi\|_\gamma \le c\,\varkappa^{1-\gamma} \|x\|_1 \quad .$$

Looking at the definition of $\varkappa_{\alpha\beta}$ we have at once

$$(3.14) \qquad \varkappa_{o1} \le c\ \varkappa$$

and a certain 'almost' best simultaneous approximability in the 0- and the γ-norm. Moreover, with the help of Lemma 2.3 we get

Corollary 3.4: Let $\varkappa$ be defined by (3.10). Then there is a $\xi \in S$ such that simultaneously

$$(3.15) \qquad \|x-\xi\|_\beta \le c\ \varkappa^{1-\beta}\ \|x\|_1$$

holds true for $0 \le \beta \le \gamma$.

An obvious (weaker) consequence is

Corollary 3.5: Let γ be given with $0 < \gamma < 1$. Then for all β with $0 \le \beta \le \gamma$

$$(3.16) \qquad \varkappa_{\beta 1} \le c\ \varkappa_{\gamma 1}^{(1-\beta)/(1-\gamma)} \quad .$$

Proof of Lemma 3.3: Let us define

$$(3.17) \qquad E(x) = \inf_{\xi \in S}\left\{\|x-\xi\|_o + \varkappa^\gamma \|x-\xi\|_\gamma\right\}$$

and

$$(3.18) \qquad \varepsilon_\delta = \sup\left\{E(x) \mid x \in H_\delta \wedge \|x\|_\delta \le 1\right\} \quad .$$

We have

$$(3.19) \qquad E(x) \le \varepsilon_\gamma\ \|x\|_\gamma \quad .$$

Since - see (3.6) -

$$(3.20) \qquad E(x) = E(x-\eta)$$

for any $\eta \in S$ this gives

$$(3.21) \qquad E(x) \le \varepsilon_\gamma\ \|x-\eta\|_\gamma \quad .$$

With the help of (3.12) we come to

$$(3.22) \qquad E(x) \le \varepsilon_\gamma\ \varkappa^{1-\gamma}\ \|x\|_1$$

and hence

$$(3.23) \qquad \varepsilon_1 \le \varepsilon_\gamma \cdot \varkappa^{1-\gamma} \quad .$$

We will need a second relation combining ε_1 and ε_γ . In order to get this we verify that the two propositions hold

Proprosition 3.6: The functional E is subadditive, i.e.

$$(3.24) \qquad E(x_1+x_2) \le E(x_1) + E(x_2) \quad .$$

This follows easily from the definition of $E(.)$, it is comparable with the triangle inequality in factor spaces.

Proposition 3.7:

$$(3.25) \qquad E(x) \le \|x\|_o + \varkappa^\gamma\ \|x\|_\gamma \quad .$$

This is obvious since $\xi = 0$ belongs to S , see the definition of $E(.)$.

The final step in the proof is done by applying Lemma 2.4: Let $x \in H_\gamma$ be given. Then there is a $y \in H_1$ according to

$$\|x-y\|_0 \le t^\gamma \|x\|_\gamma \quad ,$$

(3.26) $$\|x-y\|_\gamma \le \|x\|_\gamma \quad ,$$

$$\|x\|_1 \le t^{\gamma-1}\|x\|_\gamma \quad .$$

We get

$$E(x) \le E(x-y) + E(y)$$

(3.27) $$\le \|x-y\|_0 + \varkappa^\gamma\|x-y\|_\gamma + \varepsilon_1\|y\|_1$$

$$\le \left\{t^\gamma + \varkappa^\gamma + \varepsilon_1 t^{\gamma-1}\right\} \|x\|_\gamma$$

and

(3.28) $$\varepsilon_\gamma \le t^\gamma + \varkappa^\gamma + \varepsilon_1 t^{\gamma-1} \quad .$$

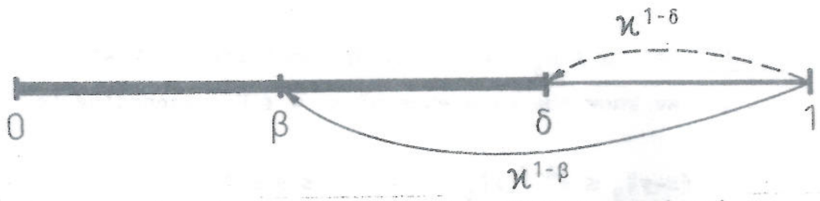
We remark that $t > 0$ is arbitrary. We replace ε_γ in (3.23) by (3.28) and come to

(3.29) $$\varepsilon_1 \le t^\gamma \varkappa^{1-\gamma} + \varkappa + \varepsilon_1(\varkappa/t)^{1-\gamma} \quad .$$

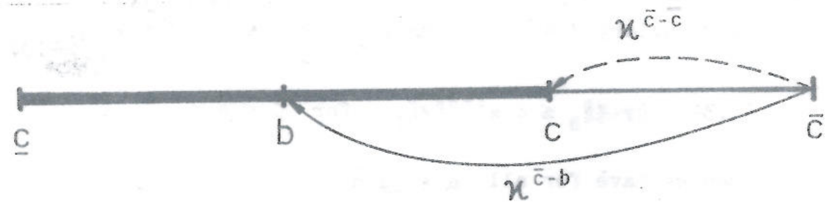
The choice $t = \varkappa\, 2^{1/(1-\gamma)}$ gives

(3.30) $$\varepsilon_1 \le \varkappa\left\{2^{\gamma/(1-\gamma)} + 1\right\} + \varepsilon_1/2 \quad .$$ #

The figure will illustrate the 'meaning' of Corollary 3.4



By rescaling we will get a corresponding approximation result for an interval $[\underline{c}, \bar{c}]$ which is given only in the graphical representation



By the lemma the simultaneous approximability in the β-norm for all $0 \le \beta \le \gamma$ of an element in H_1 is guaranteed provided the approximability in the γ-norm is known. The next lemma will show that then also bounds for the approximability of elements of the spaces H_δ with $0 < \delta < 1$ follow

<u>Lemma 3.8:</u> For $\gamma \in (0,1)$ fixed let $\varkappa$ be defined by (3.12). Further, let $\delta \in (0,1)$ and $\underline{\alpha}$, $\bar{\alpha}$ be such that

(3.31) $$0 \le \underline{\alpha} \le \bar{\alpha} \le \min\,(\gamma,\delta) \quad .$$

To any $x \in H_\delta$ there is a $\xi \in S$ according to

(3.32) $\|x-\xi\|_\alpha \le c\,\varkappa^{\delta-\alpha}\,\|x\|_\delta$ for $\alpha \in [\underline{\alpha},\bar{\alpha}]$.

<u>Proof:</u> Let $x \in H_\delta$ be given. By Corollary 2.5 - with $t = \varkappa$ - we know the existence of a $y \in H_1$ according to

$$\|x-y\|_\beta \le \varkappa^{\delta-\beta}\|x\|_\delta \quad \text{for } 0 \le \beta \le \delta ,$$

(3.33)

$$\|y\|_1 \le \varkappa^{\delta-1}\|x\|_\delta .$$

By Corollary 3.4 to y there exists a $\xi \in S$ with

(3.34) $\|y-\xi\|_\beta \le c\,\varkappa^{1-\beta}\|y\|_1$ for $0 \le \beta \le \gamma$.

Then we have for all $\alpha \in [\underline{\alpha},\bar{\alpha}]$

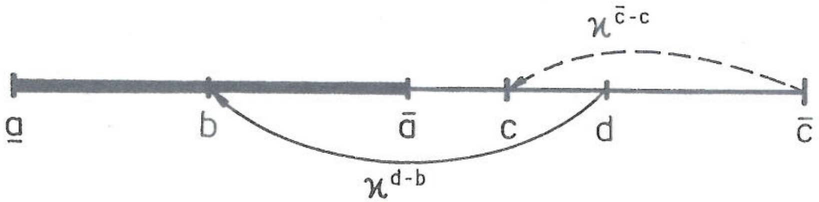
$$\|x-\xi\|_\alpha \le \|x-y\|_\alpha + \|y-\xi\|_\alpha$$

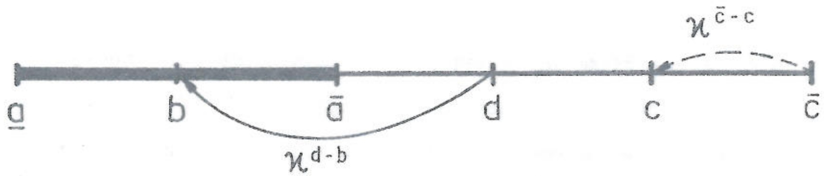
(3.35)

$$\le \varkappa^{\delta-\alpha}\|x\|_\delta + c\,\varkappa^{1-\alpha}\|y\|_1$$

$$\le (1+c)\,\varkappa^{\delta-\alpha}\|x\|_\delta .$$

Similar to above we illustrate Lemma 3.8 in case of arbitrary intervals only graphically by two figures taking into account the cases $\gamma < \delta$ and $\gamma > \delta$





The counterpart of Corollary 3.5 is the consequence of Lemma 3.8

<u>Corollary 3.9:</u> Let $\gamma,\delta \in (0,1)$ be fixed. Then for all β with $0 \le \beta \le \min(\gamma,\delta)$

(3.36) $\varkappa_{\beta\delta} \le \varkappa_{\gamma 1}^{(\delta-\beta)/(1-\gamma)}$

Up to now we have considered one approximation space S and have derived some relationships between the approximation-quantities $\{\varkappa_{\alpha\beta}\}$ for different indizes α,β . Now we will look for a sequence $\{S_n\}$ of such spaces. In the applications the dimension of S_n will be finite. For simplicity we will assume

(3.37) $\dim(S_n) = n$

with $n = 1,2,\ldots$. Of course, the modifications in case of $n = n_1,n_2,\ldots$ with $n_j \to \infty$ are obvious. We will denote by $\varkappa_{\alpha\beta}(S_n)$ the quantities $\varkappa_{\alpha\beta}$ (3.4) in case of $S := S_n$.

Within certain limits the dimension of an approximation space is the measure of work, computing time etc. needed for solving a special problem. Now let us think of two approximation spaces S^1, S^2 of the same dimension. From the viewpoint of approximation theory we will prefer for instance the space S^1 if - roughly speaking - an element x will be approximated 'better' by elements of S^1 instead of S^2. In our setting we want to compare the quantities $\varkappa_{\alpha\beta}(S_n)$ for different spaces S_n of the same dimension, and of course for different values of α, β.

In this context the 'best' or 'optimal' spaces S_n are such that - fixed the indizes α, β -

$$(3.38)\qquad \varkappa_{\alpha\beta}(S_n) = \inf_{\dim S=n} \varkappa_{\alpha\beta}(S) \quad .$$

The right hand side

$$(3.39)\qquad d^n_{\alpha\beta} := \inf_{\dim S=n} \sup_{x\in H_\beta} \inf_{\xi\in S} \|x-\xi\|_\alpha / \|x\|_\beta$$

is called the n-dimensional diameter of the unit ball of H_β in the space H_α. The formulation is self-expressing

<u>Definition 3.10:</u> Let $0 \le \alpha < \beta \le 1$ be fixed.

A sequence $\{S_n\}$ is called (α,β)-quasi-optimal if

$$(3.40)\qquad \varkappa_{\alpha\beta}(S_n) \le c\, d^n_{\alpha\beta}$$

with a constant c independent of n.

In our case of a Hilbert scale the diameters are given by

<u>Theorem 3.11:</u> Let $\{H_\alpha\}$ be a Hilbert scale as discussed in the preceeding section. Then for $\alpha \le \beta$

$$(3.41)\qquad d^n_{\alpha\beta} = \lambda_{n+1}^{(\alpha-\beta)/2}$$

<u>Proof:</u> It is to be 'expected' that the space of the n first eigen-elements

$$(3.42)\qquad E_n = sp\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

will be the optimal subspace, independent of α, β. Let P_n be the orthogonal projector onto E_n with respect to the 0-norm

$$(3.43)\qquad P_n x = \sum_1^n (x, \varphi_i)\, \varphi_i \quad .$$

For any $x \in H_\beta$ we have with $\xi = P_n x \in E_n$ - see (2.5) - and the abbreviation $x_i = (x, \varphi_i)$

$$(3.44)\qquad x - \xi = \sum_{n+1}^{\infty} x_i \varphi_i$$

and hence

$$\|x-\xi\|^2_\alpha = \sum_{n+1}^{\infty} \lambda_i^\alpha x_i^2$$

$$(3.45)\qquad \le \lambda_{n+1}^{\alpha-\beta} \sum_{n+1}^{\infty} \lambda_i^\beta x_i^2$$

$$\le \lambda_{n+1}^{\alpha-\beta} \|x\|^2_\beta \quad .$$

This inequality shows that the n-dimensional diameter is bounded by the right hand side of (3.41).

Now let S_n be a fixed subspace of H_1 with dimension n and let $\{\Psi_i | i = 1,\dots,n\}$ be a base in S_n , i.e.

$$(3.46) \qquad S_n = sp\{\Psi_1, \Psi_2, \dots, \Psi_n\} \quad .$$

In the space E_{n+1} there is an element x orthogonal to S_n with respect to the α-norm: Put $x = \Sigma x_i \varphi_i$. The orthogonality means

$$(3.47) \qquad \sum_{i=1}^{n+1} \lambda_i^\alpha x_i (\varphi_i, \Psi_j) = 0 \quad \text{for } j = 1,\dots,n \quad .$$

These conditions are n linear equations for the $n+1$ variables $\{x_i\}$. Hence there exists a nontrivial solution $\{x_i\}$ resp. an element $x \in E_{n+1}$ orthogonal to S_n with $x \neq 0$.

Because of the orthogonality we get

$$(3.48) \qquad \inf_{\xi \in S_n} \|x-\xi\|_\alpha = \|x\|_\alpha \quad .$$

But we have

$$(3.49) \qquad \begin{aligned} \|x\|_\alpha^2 &= \sum_1^{n+1} \lambda_i^\alpha x_i^2 \\ &\geq \lambda_{n+1}^{\alpha-\beta} \sum_1^{n+1} \lambda_i^\beta x_i^2 \\ &\geq \lambda_{n+1}^{\alpha-\beta} \|x\|_\beta^2 \quad . \end{aligned}$$

#

Now it is easy to show

<u>Theorem 3.12:</u> Let for $\gamma \in (0,1)$ the sequence of approximation spaces $\{S_n\}$ be $(\gamma,1)$ quasi-optimal. Then they are also (α,β) - quasi-optimal for all pairs (α,β) with $\alpha < \beta$, $\alpha \leq \gamma$, $\beta \leq 1$.

<u>Proof:</u> The $(\gamma,1)$-quasi-optimality implies

$$(3.50) \qquad \varkappa_{\gamma 1}^n := \varkappa_{\gamma 1}(S_n) \leq C \lambda_{n+1}^{(1-\gamma)/2}$$

with c independent of n . By the above lemmata we have for indices within the given range

$$(3.51) \qquad \varkappa_{\alpha\beta}^n \leq c\, C\, \lambda_{n+1}^{(\alpha-\beta)/2} \quad .$$

#

Up to now we have discussed in this section the 'approximation-quantities' $\{\varkappa_{\alpha\beta}\}$. Analogue to them 'inverse-quantities' play a vital role: Let S be a finite dimensional subspace of H_1 (similar to above we will restrict ourselves to spaces H_α with $\alpha \in [0,1]$). Since any two norms in S are equivalent - see (A.) - there are finite constants σ with

$$(3.52) \qquad \|\xi\|_\beta \leq \sigma_{\alpha.\beta} \|\xi\|_\alpha \quad \text{for } \xi \in S \quad .$$

Because of Lemma 2.2 we have

$$(3.53) \qquad \sigma_{\alpha.\beta} \leq 1 \qquad \text{for } \beta \leq \alpha \quad .$$

Therefore only the case $\alpha < \beta$ is of interest. Since the embedding $H_\beta \to H_\alpha$ is compact we expect $\sigma_{\alpha.\beta}$ tending to infinity for a sequence $\{S_n\}$ with $n \to \infty$.

In the remainder of this section we will firstly prove some relations between the σ's for different indices, then we will discuss lower limits of the σ's and finally combine the σ's with the $\varkappa$'s discussed in the first part of this section.

The counterpart of Lemmata 3.3 and 3.8 is

Lemma 3.13: Let $S \subseteq H_1$ be fixed. Further let γ with $0 < \gamma < 1$ be given. If

(3.54) $$\|\xi\|_1 \le \sigma^{-(1-\gamma)} \|\xi\|_\gamma \quad \text{for} \quad \xi \in S$$

holds true then also

(3.55) $$\|\xi\|_\beta \le \sigma^{-(\beta-\alpha)} \|\xi\|_\alpha \quad \text{for} \quad \xi \in S$$

for all pairs (α, β) according to $0 \le \alpha \le \beta \le 1$ and $\alpha \le \gamma$.

Proof: We will apply Lemma 2.3. We get for the triple $(\alpha, \gamma, 1)$

(3.56) $$\|\xi\|_\gamma^{1-\alpha} \le \|\xi\|_\alpha^{1-\gamma} \|\xi\|_1^{\gamma-\alpha} \quad .$$

With the assumption (3.54) of the lemma we get further

(3.57) $$\|\xi\|_1^{1-\alpha} \le \sigma^{-(1-\gamma)(1-\alpha)} \|\xi\|_\gamma^{1-\alpha} \le \sigma^{-(1-\gamma)(1-\alpha)} \|\xi\|_\alpha^{1-\gamma} \|\xi\|_1^{\gamma-\alpha} \quad .$$

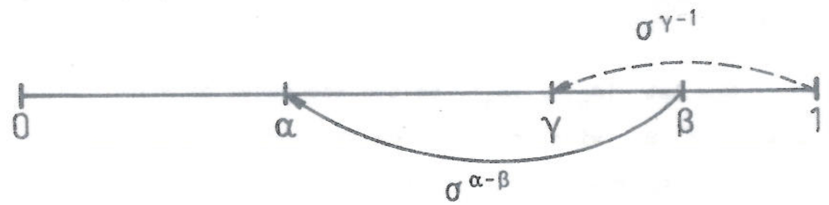
Multiplying this inequality by $\|\xi\|_1^{\alpha-\gamma}$ and taking the power $-(1-\gamma)$ gives

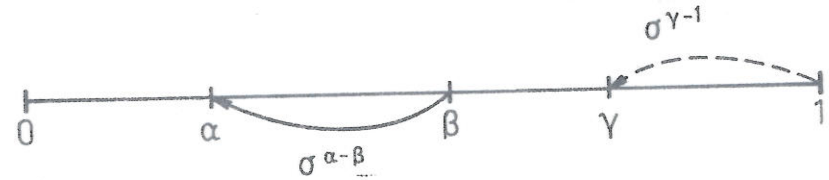
(3.58) $$\|\xi\|_1 \le \sigma^{-(1-\alpha)} \|\xi\|_\alpha \quad .$$

This is the lemma in case of $\beta = 1$. For $\alpha < \beta < 1$ we have by Lemma 2.3 and (3.58)

(3.59) $$\|\xi\|_\beta^{1-\alpha} \le \|\xi\|_\alpha^{1-\beta} \|\xi\|_1^{\beta-\alpha} \le \sigma^{-(1-\alpha)(\beta-\alpha)} \|\xi\|_\alpha^{1-\alpha} \quad . \qquad \#$$

Similar to above the two figures will illustrate Lemma 3.13.





Corresponding to (3.38,39) and Theorem 3.11 we have

<u>Theorem 3.14:</u> Let $\alpha < \beta$ and $n = \dim(S)$.

Then

$$\sigma_{\alpha.\beta} \geq \lambda_n^{(\beta-\alpha)/2} \tag{3.60}$$

<u>Proof:</u> In S there is an element orthogonal to E_{n-1} - see (3.42). Then the representation

$$\xi = \sum_n^\infty \xi_i \varphi_i \tag{3.61}$$

gives

$$\begin{aligned} \|\xi\|_\beta^2 &= \sum_n^\infty \lambda_i^\beta \xi_i^2 \\ &\geq \lambda_n^{\beta-\alpha} \sum_n^\infty \lambda_i^\alpha \xi_i^2 \\ &\geq \lambda_n^{\beta-\alpha} \|\xi\|_\alpha^2 \end{aligned} \tag{3.62}$$

which proves (3.60). In case of $S = E_n$ we have equality in (3.60). #

Now let $\{S_n\}$, $\{\tilde{S}_n\}$ be two sequences of subspaces of H_1 with n denoting the dimension of S_n and $\tilde{S}_n$. By Theorem 3.11 in connection with the definition of $d^n_{\alpha\beta}$ and Theorem 3.14 we have

$$\varkappa_{\alpha\beta}(S_n)\ \sigma_{\alpha\beta}(\tilde{S}_{n+1}) \geq 1 \quad . \tag{3.63}$$

We will give a direct proof of this inequality without using the fact that the norms belong to a Hilbert scale. Since the dimension of $\tilde{S}_{n+1}$ is greater than that of S_n there is an element $\tilde{\xi} \in \tilde{S}_{n+1}$ with $\tilde{\xi} \neq 0$ orthogonal to S_n in the metric of H_α and therefore - see (A.) -

$$\inf_{\xi \in S_n} \|\tilde{\xi}-\xi\|_\alpha = \|\tilde{\xi}\|_\alpha \quad . \tag{3.64}$$

This leads to

$$\begin{aligned} \varkappa_{\alpha\beta}(S_n) &= \sup\left\{\inf_{\xi \in S_n} \|x-\xi\|_\alpha \mid x \in H_\beta \wedge \|x\|_\beta = 1\right\} \\ &\geq \frac{\|\tilde{\xi}\|_\alpha}{\|\tilde{\xi}\|_\beta} \quad . \end{aligned} \tag{3.65}$$

resp. for this element of $\tilde{S}_{n+1}$

$$\|\tilde{\xi}\|_\beta \geq \{\varkappa_{\alpha\beta}(S_n)\}^{-1} \|\tilde{\xi}\|_\alpha \quad . \tag{3.66}$$

With the definition of $\sigma_{\alpha\beta}$ (3.52) the inequality (3.63) is shown.

Inequality (3.63) is important for the following reason: Let for one sequence $\{S_n\}$ respective $\{\tilde{S}_n\}$ be known the approximation quantities $\{\varkappa_{\alpha\beta}(S_n)\}$ resp. the inverse quantities $\{\sigma_{\alpha\beta}(\tilde{S}_n)\}$. The lower bounds for the other quantities are known by (3.63).

Our last result treats the case when the σ's and $\varkappa$'s are in 'balance':

<u>Theorem 3.15:</u> Let $\{S_n\}$, $\{\tilde{S}_n\}$ be two sequences as discussed above and assume for some pair (α, β) fixed with $\alpha < \beta$

(3.67) $$\varkappa_{\alpha\beta}(S_n)\, \sigma_{\alpha\beta}(\tilde{S}_{n+1}) \leq M$$

with a constant M independent of n . Then

(3.68) $$\varkappa_{\alpha\beta}(S_n) \leq M\, \lambda_{n+1}^{-(\beta-\alpha)/2} ,$$

i.e. the sequence $\{S_n\}$ is (α, β) - quasi-optimal.

<u>Proof:</u> (3.68) follows directly from (3.67) in connection with (3.60). The importance of Theorem 3.15 lies in the fact that the question of quasi-optimality may be answered without any explicite knowledge of the eigen-values λ .