# LEAST SQUARES APPROXIMATIONS TO FIRST ORDER ELLIPTIC SYSTEMS

by

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Summary: Linear boundary value problems for elliptic systems in the sense of Petrowski are considered. Using linear finite elements a least squares method is discussed. The concept of nearly zero boundary conditions - i.e. the boundary condition is imposed in the nodes on the boundary exactly - gives quasi-optimal error estimates in the $L_2$- and $W_2^1$-norms.

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## 1. Notations, the Analytic Problem

In the following $u, v, \dots$ will denote pairs $(u^1,u^2)$, $(v^1,v^2)$,. of functions defined in a bounded domain $\Omega \subseteq R^2$ with boundary $\partial\Omega$ sufficiently smooth. If both components are in $L_2(\Omega)$ resp. the Sobolev-spaces $W_2^k(\Omega)$ we will write $u \in H_o$ resp. $u \in H_k$. We will also use the notation

$$(u,v) = (u^1,v^1)_{L_2(\Omega)} + (u^2,v^2)_{L_2(\Omega)} ,$$

(1)

$$(u,v)_k = (u^1,v^1)_{W_2^k(\Omega)} + (u^2,v^2)_{W_2^k(\Omega)}$$

and

(2) $$\|u\| = (u,u)^{1/2} , \quad \|u\|_k = (u,u)_k^{1/2} .$$

As a model problem we will consider the elliptic system

(3) $$L\,u = f \qquad \text{in } \Omega ,$$

i.e.

(3)
$$\begin{aligned} L^1u &= u_x^1 - u_y^2 + a^{11}u^1 + a^{12}u_y^2 = f^1 \\ L^2u &= u_y^1 + u_x^2 + a^{21}u^1 + a^{22}u^2 = f^2 \end{aligned} \qquad \text{in } \Omega .$$

We remark that any elliptic system in the sense of Petrovski is - up to a coordinate-transformation-equivalent to (3), see

HAACK-HELLWIG [ 1 ] . In addition to (3) we impose the boundary condition

(4) $$l^1(u) = u^1 \cos \sigma + u^2 \sin \sigma = 0 \qquad \text{on } \partial\Omega .$$

Essential for the solvability of the boundary value problem is the index of $\sigma = \sigma(s)$ - s arc-length on $\partial\Omega$ - defined by

(5) $$\text{ind}\ (\sigma) = \frac{1}{2\pi} \oint_{\partial\Omega} \partial\sigma .$$

In case of $n = \text{ind}\ (\sigma) \geq 0$ then (3), (4) is always solvable, the number of linear independent solutions of the homogeneous problem $(f = 0)$ is $2n + 1$ . In case of $n < 0$ (3), (4) possesses a solution only if $f$ fulfills $2|n|-1$ linear independent integral relations.

In order to characterize in case of $n \geq 0$ a special solution to following way is possible: Let $\mathfrak{B}_n(\partial\Omega)$ be a subspace of $L_2(\partial\Omega)$ of dimension $2n + 1$ similar to the space of trigonometric functions of order $n$ . This means there is up to a factor exactly one element in $\mathfrak{B}_n(\partial\Omega)$ with zeros in $2n$ prescribed points on $\partial\Omega$ , not necessarily distinct. Then there is one and only one solution of (3), (4) such that

(6) $$\oint_{\partial\Omega} p_i\, l^2(u)\, ds = r_i \qquad (i = 1,\ldots,2n+1) .$$

Here

(7) $$l^2(u) = -u^1 \sin \sigma + u^2 \cos \sigma$$

denotes the orthogonal complement of the boundary condition (4), the set $\{p_i\}$ forms a basis of $\mathfrak{B}_n(\partial\Omega)$ and $\{r_i\}$ are fixed real numbers.

In case of a non-negative index $n$ there are especially $2n + 1$ solutions $u_j$ of (3), (4) with $f = 0$ according to

(8) $$\oint_{\partial\Omega} p_i\, l^2(u_j) ds = \delta_{ij} .$$

For negative indices $n = \text{ind}\ (\sigma)$ on the other hand there exists one and only one solution of (3) such that

(9) $$l^1(u) \in \mathfrak{P}_{2|n|-1} .$$

In the following we will impose the conditions (6) for non-negative indices of $\sigma$ and use the weakened form (9) for negative indices.

For functions $p,q \in L_2(\partial\Omega)$ we will use

(10) $$\langle p,q \rangle = \oint p\, q\, ds ,$$
$$|p| = \langle p,p \rangle^{1/2} .$$

The conditions (6) may be rewritten:

(6') $$\langle p_i, l^2(u) \rangle = r_i \qquad (i = 1,\ldots,2n+1) .$$

For $n = \text{ind}\ (\sigma) \geq 0$ - only this case will be considered here - let $H_1^\sigma$ be the set of pairs $u = (u^1,u^2)$ with $u \in H_1$ and fulfilling (4) and $H_1^{\sigma \cdot r}$ the subspace fulfilling (6).

For the sake of simplicity the coefficients $a^{ij}$ in (3) as well as $\sigma$ in (4) are assumed sufficiently smooth. Without loss of generality the elements of $\mathfrak{P}_h(\partial\Omega)$ can be chosen sufficiently smooth and therefore $\mathfrak{P}_h(\partial\Omega)$ can be extended to a space $\mathfrak{P}_h(\bar{\Omega})$ of functions defined in $\bar{\Omega}$ .

For the modified boundary vlaue problems (3), (4), (6) in case of a non-negative index resp. (3), (9) in case of a negative index shift-theorems of the type

(11) $$\|u\|_{k+1} \leq c \left\{ \|f\|_k + \Sigma \, |r_i| \right\}$$

are valid. c depends besides of $\sigma$ , $a^{ik}$ on $\partial\Omega$ and $k$ .

The statements of these sections are consequences of the theory on elliptic systems developed by VEKUA [2] .



## 2. Finite Element Spaces

Let $\Gamma_h$ be a subdivision of $\Omega$ into generalized triangles $\Delta$ , i.e. $\Delta$ is a triangle if $\bar{\Delta}$ and $\partial\Omega$ have in common at most one point and otherwise one of the sides of $\Delta$ may be curved. We will only consider regular subdivisions: For $\varkappa > 1$ fixed there are to any $\Delta \in \Gamma_h$ two circles with radii $\varkappa^{-1}h$ and $\varkappa h$ contained in $\Delta$ resp. containing $\Delta$ .

The finite element spaces $S_h = S_h(\Gamma_h)$ we will work with consist of pairs $\chi = (\chi^1, \chi^2)$ of functions having the properties

i) $\chi \in H_1$ , i.e. $\chi^i \in W_2^1(\Omega)$ ,

ii) $\chi^i$ restricted to any $\Delta \in \Gamma_h$ is linear.

Let $\{P_\nu\}$ be the set of nodes of $\Gamma_h$ on $\partial\Omega$ . The subspace $S_h^\sigma \subseteq S_h$ consists of those elements with

(12) $$l^1(\chi)_{|P_\nu} = 0 \quad .$$

Finally $S_h^{\sigma \cdot r} \subseteq S_h^\sigma$ consists of elements with

(13) $$< p_i \, , \, l^2(\chi) > \, = r_i \qquad (i = 1, \ldots, 2n+1) \; .$$

Then the following approximation property holds:

<u>Lemma:</u> Let $n = \text{ind}\,(\sigma) \geq 0$ . There is a linear projection-operator $Q_h = Q_h^{\sigma \cdot r} : H_1^{\sigma \cdot r} \to S_h^{\sigma \cdot r}$

with

(14) $\|u-Q_h u\|_1 \le c\ h\ \|u\|_2$

for $u \in H_2^{\sigma \cdot r} = H_1^{\sigma \cdot r} \cap H_2$ .

First let $I_h$ denote the linear interpolation. For $u \in H_2^\sigma$ we have obviously $I_h u \in S_h^\sigma$ and

(15) $\|u-I_h u\|_1 \le c\ h^{2-l}\|u\|_2 \qquad (l = 0,1)$ .

Further we get

(16) $|r_i - < p_i\ ,\ l^2(I_h u) > | \le |p_i|\ |l^2(u-I_h u)|$

and because of

(17) $|z|^2 \le \|z\|_{L_2(\Omega)}\ \|z\|_{W_2^1(\Omega)}$

for any $z \in W_2^1(\Omega)$ therefore

(18) $|r_i - < p_i\ ,\ l^2(I_h u) > | \le c\ h^{3/2}\ \|u\|_2$ .

In order to get $Q_h$ we add a proper combination of the interpolated homogeneous solutions of (8)

(19) $Q_h u = I_h u + \sum_{i=1}^{2n+1} \alpha_i\ I_h u_i$ .



The conditions on $\{\alpha_i\}$ are

(20) $\sum_{i=1}^{2n+1} < p_j\ ,\ l^2(I_h u_i) > \alpha_i = < p_j,\ l^2(u-I_h u) >$

$(j = 1,\ldots,2n+1)$ .

Because of

(21) $|l^2(u_i-I_h u_i)| \le c\ h^{3/2}$

and (8) the inverse of the matrix of the linear equations (20) is bounded away from zero for $h$ small enough. Using the bound (18) for the right hand side of (20) we get

(22) $|\alpha_i| \le c\ h^{3/2}\ \|u\|_2$ .

Therefore

(23) $\|u-Q_h u\|_1 \le \|u-I_h u\|_1 + \mathrm{Max}\left\{\|I_h u_i\|_1\right\} \sum_{1}^{2n+1} |\alpha_i|$

which gives (14) .

## 3. Least Squares Method, Error Estimates

For $u \in H_1$ and $v \in H_1$ resp. $w \in H_o$ let us define the bilinear functionals

$$a(u,v) = (Lu, Lv) = (L^1u, L^1v) + (L^2u, L^2v) \tag{24}$$

resp.

$$b(u,w) = (Lu, w) = (L^1u, w^1) + (L^2u, w^2) \quad . \tag{25}$$

Obviously we have

$$a(u,v) = b(u, Lv) \quad . \tag{26}$$

We get by partial integration - for $w \in H_1$ -

$$b(u,w) = (u, {}^{*}Lw) + \oint l^1(u)\{w^1 \sin(\sigma+\gamma) - w^2 \cos(\sigma+\gamma)\}ds + \oint l^2(u)\{w^1 \cos(\sigma+\gamma) + w^2 \sin(\sigma+\gamma)\}ds$$

with $\gamma$ denoting the angle between the tangent at a point of $\partial\Omega$ and the x-axis, and



$$\begin{aligned} {}^{*}L^1w &= -w^1_x - w^2_y + a^{11}w^1 + a^{21}w^2 \quad , \\ {}^{*}L^2w &= w^1_y - w^2_x + a^{12}w^1 + a^{22}w^2 \end{aligned} \tag{28}$$

being the adjoint of the differential operator $L$ .

If

$$n = \text{ind}\,(\sigma) \geq 0 \tag{29}$$

then the index of the boundary condition

$$w^1 \cos(\sigma+\gamma) + w^2 \sin(\sigma+\gamma) = 0 \tag{30}$$

with respect to the operator ${}^{*}L$ - $w^1$ and $w^2$ have to be interchanged in order to give the Cauchy-Riemann principle part - is

$$\begin{aligned} n^* &= \text{ind}\,(\sigma^*) \\ &= \text{ind}\,(\tfrac{\pi}{2}-\sigma-\gamma) = -n - 1 \quad . \end{aligned} \tag{31}$$

According to (9) then (30) has to be modified

$$w^1 \cos(\sigma+\gamma) + w^2 \sin(\sigma+\gamma) \in P_{2n+1} \qquad \text{on } \partial\Omega \tag{32}$$

in order that

$${}^{*}Lw = g \qquad \text{in } \Omega \tag{33}$$

together with (32) has a unique solution.

For simplicity we will consider the boundary value problem (3), (4), (6) only with $r_i = 0$ . Then we have the duality relation

(34) $$(u,g) = b(u,w) \quad .$$

Further let $v$ be defined by

(35) $$\begin{aligned} L\ v &= w && \text{in } \Omega \ , \\ l^1(v) &= 0 && \text{on } \partial\Omega \ , \\ < p,\ l^2(v) > &= 0 && \text{for } p \in P_{2n+1} \ . \end{aligned}$$

Then we have

(36) $$(u,g) = a(u,v) \quad .$$

Using the shift theorem (11) we get $v \in H_{k+2}$ for $g \in H_k'$ .

In order to approximate the solution $u$ of (3), (4), (6) - with $r_i = 0$ - we use the least squares method: The approximation $u_h \in S_h^{\sigma,0}$ is defined by

(37) $$a(u_h,\chi) = (f,L\chi) \qquad \text{for } \chi \in S_h^{\sigma,0} \ .$$

Though $a(.,.)$ is positive definite in $H_1^{\sigma,0}$ it might only be semi-definite in $S_h^{\sigma,0}$ . With $e = u - u_h$ and - using an appropriate approximation $U_h$ on $u$ in $S_h^{\sigma,0}$ - $\varepsilon = u-U_h$ and therefore $e = \varepsilon + \phi$ with $\phi = U_h - u_h \in S_h^{\sigma,0}$ we get



(38) $$a(e,\chi) = 0 \qquad \text{for } \chi \in S_h^{\sigma,0}$$

resp.

(39) $$a(\phi,\chi) = -a(\varepsilon,\chi) \qquad \text{for } \chi \in S_h^{\sigma,0} \ .$$

By

(40) $$\|\cdot\|' = a(.,.)^{1/2}$$

a semi-norm is defined. Obviously we have for $v \in H_1$

(41) $$\|v\|' \leq c\ \|v\|_1 \quad .$$

So we get from (39)

(42) $$\begin{aligned} \|\phi\|' &\leq \|\varepsilon\|' \\ &\leq c\ \|\varepsilon\|_1 \leq c\ h\ \|u\|_2 \end{aligned}$$

and consequently

(43) $$\|e\|' \leq 2\ c\ h\ \|u\|_2 \quad .$$

Next we identify $g = e$ in (34) and let $w$ resp. $v$ be the solutions of (32), (33) resp. (35). Then we get

$$\|e\|^2 = (e, {}^*Lw)$$

(44)

$$= (Le, w) - \oint \{l^1(e)\ {}^*l^2(w) - l^2(e)\ {}^*l^1(w)\}\ ds \quad .$$

Because of (13) and (32) the last term on the right hand side vanishes. Therefore we get

(45) $$\|e\|^2 = (Le, Lv) - \oint l^1(e)\ {}^*l^2(w)\ ds \quad .$$

We will estimate the two terms separately. Using the shift-theorem (11) we find $v \in H_2$ and $\|v\|_2 \leq c\|e\|$. With an appropriate approximation $\chi \in S_h^{\sigma \cdot o}$ on $v$ we get with (38), (43)

(46)
$$\begin{aligned}(Le, Lv) &= a(e, v-\chi)\\ &\leq \|e\|'\ c\ h\|v\|_2\\ &\leq c\ h^2\|u\|_2\ \|e\| \quad .\end{aligned}$$

In order to find a bound for the second term in (45) we first notice - using (17) -

(47) $$|{}^*l^2(w)| \leq c\|e\| \quad .$$

Next we make use of conditions (12) which play the role of 'nearly zero boundary conditions' introduced in [2] . With arguments parallel to there we get

<u>Proposition 1:</u> To any $v \in H_1^{\sigma \cdot o} \cap H_2$ there is a $\chi \in S_h^{\sigma \cdot o}$ according to

(48) $$|l^1(v-\chi)| \leq c\ h^2\|v\|_2 \quad .$$



<u>Proposition 2:</u> For any $\chi \in S_h^{\sigma \cdot o}$ nearly zero boundary conditions of the type

(49) $$|l^1(\chi)| \leq c\ h^{3/2}\ \|\chi\|_1$$

hold.

For $\varkappa$-regular triangulations inverse properties

(50) $$\|\chi\|_1 \leq c\ h^{-1}\ \|\chi\|$$

hold for $\chi \in S_h$ . Therefore (49) can be replaced by

(51) $$|l^1(\chi)| \leq c\ h^{1/2}\ \|\chi\| \quad .$$

Now let $U_h$ be an approximation on $u$ according to Proposition 1 and put $\phi = U_h - u_h \in S_h^{\sigma \cdot o}$ . Then we get

(52)
$$\begin{aligned}|l^1(e)| &\leq |l^1(u-U_h)| + |l^1(\phi)|\\ &\leq c\ h^2\|u\|_2 + c\ h^{1/2}\ \|\phi\|\\ &\leq c\ h^2\|u\|_2 + c\ h^{1/2}\{\|u-U_h\| + \|u-u_h\|\}\\ &\leq c\ h^2\|u\|_2 + c\ h^{1/2}\|e\| \quad .\end{aligned}$$

The bounds (46) and (47), (52) give - see (45) -

$$(53) \qquad \|e\|^2 \leq c\ h^2 \|u\|_2\ \|e\| + c\ h^{1/2} \|e\|^2 .$$

and so for $h$ small enough

$$(54) \qquad \|e\| \leq c\ h^2 \|u\|_2 \quad .$$

Since we now know $e$ to be bounded we have as a consequence the unique solvability of the defining equations (37).

Using (50) we also get the error estimate

$$(55) \qquad \|e\|_1 \leq c\ h \|u\|_2 \quad .$$

## Literature


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