A FINITE ELEMENT METHOD FOR PARABOLIC FREE BOUNDARY PROBLEMS

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0. Introduction

In [9] a semi-discrete finite element method for the one-phase Stefan problem in one space dimension was proposed and analized. In [10,11] the error analysis was further extended to the case of irregular initial data such that the 'oxygen diffusion problem' - see Problem I in MAGENES [8] - is covered.

After summarizing these results in Section 1 we discuss in Section 2 some refinements and further generalizations, especially Problem III in MAGENES [8] fits into this framework.

Primarily the method was developed in order to solve numerically parabolic free boundary problems. On the other hand it can be used for the proof of existence and uniqueness. This is shown in Section 3. We remark that the results are known in principle. The main feature of this approach is the elementary way of getting the a priori estimates needed.

There is an extensive literature on free boundary problems both with respect to theoretical and numerical aspects. We have tried to give a representative list of papers concerning the parabolic case in the Bibliography B .

1. The Finite Element Method

The Stefan problem in its simplest form is as follows: A pair of functions $\{U = U(y,\tau),\ s = s(\tau)\}$ is sought such that U solves the heat equation

$$(1.1) \qquad U_\tau - U_{yy} = 0$$

in the domain

$$(1.2) \qquad \Omega = \left\{(y,\tau) \mid \tau > 0 \wedge 0 < y < s(\tau)\right\} .$$

The initial data are given: $U(y,0) = f(y)$. At the fixed boundary $y = 0$ the flux is zero: $U_y(0,\tau) = 0$. Along the free boundary $y = s(\tau)$ the temperature is zero: $U(s(\tau),\tau) = 0$. The 'moving' of it is governed by

$$(1.3) \qquad s_\tau + U_y(s(\tau),\tau) = 0 \quad , \qquad s(0) = 1 \quad .$$

By introducing $x = s^{-1}y$ as new space variable the problem is transformed to one with a fixed boundary. It is usefull to introduce a new time-variable t by means of

$$(1.4) \qquad t = \int_0^\tau s^{-2}(\tau)\, d\tau \quad .$$

Of course, if s is known as a function of t then τ is

defined by

$$(1.5) \qquad \frac{d\tau}{dt} = s^2(t), \ \tau(0) = 0 \quad .$$

The function $u = u(x,t) = U(y,\tau)$ solves

<u>Problem</u> P_u: <u>Find</u> u <u>such that</u>

$$(1.6) \qquad u_{xx} - u_t = xu_x(1,t)u_x \ \underline{\text{in}}\ Q = \left\{(x,t) \mid 0 < x < 1 \wedge t>0\right\} ,$$

$$(1.7) \qquad u_x(0,t) = 0$$

$$(1.8) \qquad u(1,t) = 0 \qquad \Big\}\ \underline{\text{for}}\ t > 0 \quad ,$$

$$(1.9) \qquad u(x,0) = f(x) \ \ \underline{\text{for}}\ \ 0 < x < 1 \quad .$$

In our case - vanishing flux at $x = 0$ - the free boundary is not coupled with Problem P_u . Once u is known then s is determined by

$$(1.10) \qquad \frac{ds}{dt} = - u_x(1,t)s \ , \ s(0) = 1 \quad .$$

Because of (1.8) u can be computed if $v = u_x$ is known, which for t fixed has to be in the space

$$(1.11) \qquad \dot{H}_1 = \left\{w \mid w \in H_1(0,1) \wedge w(0) = 0\right\} \quad .$$

Multiplication of (1.6) by w_x with $w \in \dot{H}_1$ and integration with respect to x leads to a 'weak formulation' of Problem P_u in terms of v :

<u>Problem</u> P_v : <u>Find</u> v <u>with</u> $v(.,t) \in \dot{H}_1$ <u>such that</u>

$$(1.12) \qquad (\dot{v},w) + (v',w') = v(1)(xv,w')$$

$$\underline{\text{for}}\ \ w \in \dot{H}_1 \ \ \underline{\text{and}}\ \ t > 0$$

<u>with the initial data</u>

(1.13) $v(.,0) = g := f'$.

Here and in the following v' and $\dot{v}$ denote differentiation with respect to x and t, sometimes we will also write $\dot{v} = \partial_t v$. The dependence on t is mostly suppressed, $v(1)$ means $v(1,.)$. The L_2-product is denoted by $(.,.)$ and the norm by $\|.\|$.

Having this formulation in mind a finite element method is obvious:

<u>Problem</u> P_{v_h}: <u>Let</u> $S_h \subseteq \dot{H}_1$ <u>be an approximation space</u>. <u>Find</u> v_h <u>with</u> $v_h(.,t) \in S_h$ <u>such that</u>

(1.14) $(\dot{v}_h, \chi) + (v_h', \chi') = v_h(1)(xv_h, \chi')$

<u>for</u> $\chi \in S_h$ <u>and</u> $t > 0$

<u>with the initial data</u>

(1.15) $v_h(.,0) = P_h g$.

For simplicity we will consider only the case of P_h being the L_2-projector onto S_h .

Once v_h is known approximations on u resp. s, τ are given by

(1.16) $u_h' = v_h$, $u_h(1) = 0$

respective

(1.17) $\dot{s}_h = -v_h s_h$, $s_h(0) = 1$,

(1.18) $\dot{\tau}_h = s_h^2$, $\tau_h(0) = 0$.

It was proved in [9] - see also Section 4 - that Problem P_{v_h} admits locally in time a unique solution, i.e. there is a $T > 0$ depending only on $\|g\|$ and especially not on the choice of S_h such that there exists a unique solution for $0 \leq t \leq T$.

With the help of (1.12) and (1.14) we get the following relation for the error $e = e_h = u - u_h$

(1.19) $(\dot{e}, \chi) + (e', \chi') - v(1)(xe, \chi') - e(1)(xv, \chi') =$

$= -e(1)(xe, \chi')$ for $\chi \in S_h$ and $t > 0$.

With v being the solution of Problem P_v and fixed the bilinear form

(1.20) $a(e, \chi) := (e', \chi') - v(1)(xe, \chi') - e(1)(xv, \chi')$

is bounded in $\dot{H}_1$ and admits a Gårding-type inequality. Therefore standard arguments in the error analysis are applicable. In the case of a 'regular' solution optimal order of convergence is valid - see [9]. If for instance only $g \in L_2$ is assumed then even $\|v'\|$ and hence $|v|$ is not necessarily bounded for $t \to 0$. This case is discussed in [10]. The oxygen-diffusion-problem leads for the standard initial data

(1.21) $\tilde{f}(x) = \frac{1}{2}(1-x)^2$

to the initial data of v

(1.22) $$g = -\delta'(x)$$

with δ being the Dirac function - the time-derivative $U = \tilde{U}_\tau$ of the former solution solves the Stefan problem. This special situation is discussed in [11]. Of course in this case the singularity at $x = t = 0$ is known. In practice one would take into account this fact and modify the data. It is of interest that even linear splines used with the initial data defined by (1.15) lead to an h^2 convergence.

2. Refinements and Generalizations

i.) The original unknown u is approximated by - see (1.16)

(2.1) $$u_h = -\int_x^1 v_h \, dx \quad .$$

Now let $S_h = S_h^{1.r}$ be the space of C^0-splines of degree r, i.e. the elements of S_h are continuous functions which are piecewise polynomials of degree less than r. Then u_h belongs to the corresponding space $S_h^{2.r+1}$. An L_2-estimate of $u-u_h$ is equivalent to an estimate of e in the norm of H_{-1}. It can be shown that for $r \geq 3$ the error $\|u-u_h\|$ is bounded up to a constant by the error $\|e\|$ times an extra factor h. The proof follows the lines of either THOMEE [12] or ARNOLD-DOUGLAS [1].

ii.) In applying the finite element method to the oxygen diffusion problem the function U in our context is the time derivative $\partial_\tau \tilde{U}$ of the solution of the original

problem which solves

(2.2) $$\tilde{U}_\tau - \tilde{U}_{yy} = -1 \quad .$$

In this way an approximation $\tilde{u}_h$ on $\tilde{u}$ may be defined by

(2.3) $$\partial_x^2 \tilde{u}_h = s^2(1+u_h) \quad \text{in} \quad 0 < x < 1$$
$$\tilde{u}_{h|x}(0) = \tilde{u}_h(1) = 0 \quad .$$

Now in the same way using the mentioned super-convergence results respective negative norm estimates an additional factor $c\,h^2$ is won for the error of $\tilde{u}-\tilde{u}_h$. Because of duality then $r \geq 5$ is needed, i.e. at least quartic splines are necessary.

iii.) The free boundary is approximated by s_h(1.17). The error of $s-s_h$ is of the same order as $v(1)-v_h(1)$. Since $x = 1$ is always a knot a super-convergence order $h^{2(r-1)}$ holds. For this we refer to the already mentioned papers and to DOUGLAS [2], DOUGLAS-DUPONT [3], DOUGLAS-DUPONT-WHEELER [4,6], and WHEELER [13].

iv.) If the flux at $y = 0$ is not identical zero then (1.7) has to be replaced by the condition

(2.4) $$u_x(0) = \Phi(\tau(t))\, s$$

with $\Phi(\tau)$ given. The variational formulation (1.12) is still valid but the condition '$v \in \dot{H}_1$' has to be replaced by '$v-(1-x)\,\Phi(\tau(t)s \in \dot{H}_1'$. In this way (1.12) is coupled with the two ordinary differential equations (1.5), (1.10) with u_x replaced by v . The corresponding error analysis follows the same lines with qualitatively the same bounds.

<u>v.)</u> Sometimes instead of the flux the temperature U is prescribed. For simplicity let us assume

$$(2.5) \qquad U(0,t) = -k \quad ,$$

for time dependent data modifications similar to <u>iv.)</u> are necessary. In this case, of course, we cannot work with Problem P_v . Multiplication of (1.6) by w'' with

$$(2.6) \quad w \in \dot{H}_2 = \overset{o}{H}_1 \cap H_2 = \{z \mid z \in H_2(0,1) \wedge z(0) = z(1) = 0\}$$

gives after integration the counterpart of (1.12)

$$(2.7) \quad (\dot{u}',w') + (u'',w'') = u'(1)(xu',w'') \quad .$$

The corresponding finite element method was analyzed for linear problems by DOUGLAS-DUPONT-WHEELER [5]. Since now second derivatives enter at least cubic splines have to be used in order to get optimal error estimates in the L_2- resp. L_∞-norm.

<u>vi.)</u> The condition at the free boundary in case of the mentioned Problem III in MAGENES [8] is

$$(2.8) \qquad U(s(\tau),\tau) = s(\tau)$$

or equivalently

$$(2.9) \qquad U_\tau = U_y^2 - U_y \quad \text{for} \quad \tau > 0 \quad \text{and} \quad y = s(\tau) \quad .$$

This leads to the condition for u

$$(2.10) \qquad u_t = -su_x \qquad \text{for} \quad x = 1,\ t > 0 \quad .$$

For the moment let us assume zero flux at $y = 0$. Similar to Problem P_v we come now to the weak formulation for $v = u_x$.

<u>Problem P_v': Find</u> v <u>with</u> $v(.,t) \in \dot{H}_1$ <u>and</u> s <u>such that</u>

$$(2.11) \quad (\dot{v},w) + (v',w') + s\,v(1)\,w(1) = v(1)(xv,w') \qquad \underline{\text{for}} \quad w \in \dot{H}_1 \quad \underline{\text{and}} \quad t > 0 \quad ,$$

$$(2.12) \qquad \dot{s} = -vs \qquad \underline{\text{for}} \quad t > 0$$

<u>with the initial data</u>

$$(2.12) \qquad v(.,0) = g \quad ,$$

$$(2.14) \qquad s(0) = 1 \quad .$$

With the help of v,s then u is defined by

$$(2.15) \qquad u = s - \int_x^1 v\,dx \quad .$$

The modifications described in <u>iv.)</u> take place in case $v(0,t) \neq 0$. The analysis of P_v' is even simpler than of P_v because of the additional term in (2.11).

<u>3. A Priori Estimates</u>

In this section we turn over to the 'application' of the finite element method in order to derive existence and regularity results for the Stefan problem. The main tool is

Theorem: Consider Problem P_{v_h} with the assumed regularity $g \in L_2(0,1)$ of the initial data. Further let $S_h \subseteq \dot{H}_1$ be a finite dimensional approximation space. There is a $T > 0$ depending only on $\|g\|$ such that P_{v_h} has a unique solution for $t \leq T$. The semi-discrete Galerkin-approximation v_h is in $C^\infty((0,T), \dot{H}_1(0,1))$ and a priori bounds of the type

$$(3.1)\qquad \begin{aligned} &\sup_{0\le t\le T}\left\{t^{2\nu}\|\partial_t^\nu v_h\|^2 + \int_0^t \sigma^{2\nu}\|\partial_t^\nu v_h'\|^2 d\sigma\right\} \leq \gamma_{2\nu}^2 \ , \\ &\sup_{0\le t\le T}\left\{t^{2\nu+1}\|\partial_t^\nu v_h'\|^2 + \int_0^t \sigma^{2\nu+1}\|\partial_t^{\nu+1} v_h\|^2 d\sigma\right\} \leq \gamma_{2\nu+1}^2 \end{aligned}$$

are valid. The γ's are independent of S_h and

$$(3.2)\qquad \bar{\gamma}_\nu = \mathrm{Max}(\gamma_{2\nu}, \gamma_{2\nu+1})$$

is bounded by

$$(3.3)\qquad \bar{\gamma}_\nu \leq K^\nu(\nu!)^2$$

with K depending only on $\|g\|$.

The proof of the theorem which was announced in [10] is given in Section 4 . Here we will discuss some modifications and consequences

a.) If g has a higher regularity and parallel to this fulfills the corresponding compatibility conditions a priori estimates of the above type but with lower powers of t are valid.

b.) Because of

$$(3.4)\qquad z(x) = \int_0^x z'd\xi, \quad z^2(x) = 2\int_0^x zz'd\xi$$

for any $z \in \dot{H}_1$ the estimates

$$(3.5)\qquad \|z\| \leq \|z'\|$$

and

$$(3.6)\qquad |z| = \sup_{0\le x\le 1} |z(x)| \leq \sqrt{2}\, \|z\|^{1/2}\|z'\|^{1/2}$$

hold true. We will make use of this extensively in Section 4. Because of (3.1) we get especially

$$(3.7)\qquad |v_h(\cdot,t)| \leq \sqrt{2}\, \bar{\gamma}_0\, t^{-1/4} \quad .$$

Since the approximation s_h on the free boundary is defined by (1.17) a uniform Hölder-continuity of s_h with an exponent up to 3/4 is the consequence.

c.) In Section 2, vi.) we proposed a finite element method for Problem III in MAGENES [8]. By quite the same arguments existence and regularity properties of the solution of P_ν' are derived. Especially also in this case the free boundary is C^∞ .

d.) Now let $\{S_n = S_{h_n} | n = 1,2,\ldots\}$ be a nested set of approximation spaces dense in $\dot{H}_1$, i.e. $S_n \subseteq S_{n+1}$ and

$$(3.8)\qquad \lim_{n\to\infty} \sup\left\{\inf_{\xi\in S_{h_n}} \|z'-\xi'\|\right\} = 0$$

for $z \in \dot{H}_1$. Because of the a priori estimates there is at least a subsequence $v_{n'} \in S_{n'}$ converging to an element $v \in C^\infty((0,T), H_1(0,1))$ being a solution of Problem P_v and admitting the same bounds. The arguments are standard, see LIONS [7], pp 9-14.

Concerning C^∞-results of the one-phase, one dimensional Stefan problem we refer especially to CAFFARELLI-RIVIERE (1976), CANNON-HILL (1968), FRIEDMAN (1976a), SCHÄFFER (1976) - see also KINDERLEHRER (1978b), and KINDERLEHRER-NIERENBERG (1978 a,b) for the more-dimensional case.

4. Proof of the Theorem

In the first two steps explicit bounds for γ_o, γ_1 are derived. It is of importance that only then a restriction on $T = T(\|g\|)$ is necessary. In Step 3 a bound for $\gamma_{2\nu}$ in terms of $\bar{\gamma}_\mu$ with $\mu \le \nu - 1$ is given and finally in Step 4 $\gamma_{2\nu+1}$ is bounded by $\gamma_{2\nu}$ and $\bar{\gamma}_\mu$. The method is elementary in the sense that only norm estimates, Schwarz' and Young's inequality and partial integration are needed. Of course at the end the formulae are lengthy.

In the following S_h and hence v_h is fixed. Since P_{v_h} leads to a system of ordinary differential equation with a quadratic right hand side v_h exists in some neighborhood of $t = 0$. Throughout we will use the abbreviations

(4.1) $$z := v_h \ , \ z_\nu := \partial_t^\nu v_h$$

and

(4.2) $$\int t^{2\nu}\|z_\nu\|^2 := \int_0^t \sigma^{2\nu}\|z_\nu(\cdot,\sigma)\|^2 \, d\sigma \quad .$$

We note

(4.3) $$\|z_{t=0}\| = \|P_h g\| \le \|g\| \quad .$$

For the sake of clarity numerical constants are denoted by $c_1, c_2, \ldots$. If there is a dependency on the (fixed) quantity $\|g\|$ we will write $k_1, k_2, \ldots$.

<u>**LEMMA 1:**</u> <u>There is a</u> $T_1 = T_1(\|g\|)$ <u>such that for</u> $t \le T_1$

(4.4) $$\|z\|^2 + \int \|z'\|^2 \le 2\|g\|^2 \quad .$$

<u>Proof:</u> The choice $\chi = z$ in (1.14) gives

(4.5) $$\tfrac{1}{2}\partial_t \|z\|^2 + \|z'\|^2 \le |z|\,\|z\|\,\|z'\| \quad .$$

By (3.6) and Young's inequality we get

(4.6) $$|z|\,\|z\|\,\|z'\| \le \sqrt{2}\ \|z\|^{3/2}\|z'\|^{3/2} \le \tfrac{1}{2}\|z'\|^2 + \tfrac{1}{2}c_1\|z\|^6$$

and in this way by integrating (4.5) and using (4.3)

(4.7) $$\|z\|^2 + \int \|z'\|^2 \le \|g\|^2 + c_1 \int \|z\|^6 \quad .$$

The solution λ of the integral equation $\lambda = \|z\|^2$

(4.8) $$\lambda = \|g\|^2 + c_1 \int \lambda^3 \quad ,$$

i.e. the function

(4.9) $$\lambda = \lambda(t) = \|g\|^2(1-2c_1\|g\|^4 t)^{-1/2}$$

is a bound of the left hand side of (4.7). The choice

(4.10) $$T_1 = 3/(8c_1\|g\|^4)$$

leads to (4.4).

LEMMA 2: There is a $T = T(\|g\|) \le T_1$ such that for $t \le T$

(4.11) $$t\|z'\|^2 + \int t\|\dot z\|^2 \le k = k(\|g\|) \quad .$$

Proof: Now we take $\chi = \dot z$ in (1.14) and get

(4.12) $$\|\dot z\|^2 + \tfrac{1}{2}\partial_t\|z'\|^2 = z(1)(xz,\dot z') =: A_1 \quad .$$

Since $\dot z'$ enters the right hand side which is not covered by terms on the left hand side we have to integrate by parts:

$$A_1 = z^2(1)\dot z(1) - z(1)(xz'+z,\dot z)$$

(4.13) $$\le \tfrac{1}{3}\partial_t\ z^3(1) + 2\sqrt{2}\ \|z\|^{1/2}\|z'\|^{3/2}\|\dot z\|$$

$$\le \tfrac{1}{3}\partial_t\ z^3(1) + \tfrac{3}{4}\|\dot z\|^2 + c_2\|z\|\|z'\|^3 \quad .$$

Because of Lemma 1 we get from (4.12)

(4.14) $$\tfrac{1}{2}\|\dot z\|^2 + \partial_t\|z'\|^2 \le \tfrac{2}{3}\partial_t\ z^3(1) + 4\ c_2\|g\|\|z'\|^3$$

and hence by multiplication with t and integration

(4.15) $$t\|z'\|^2 + \tfrac{1}{2}\int t\|\dot z\|^2 \le \tfrac{2}{3}t\ z^3(1) + \int\|z'\|^2 - \tfrac{2}{3}\int z^3(1) + 4\ c_2\|g\|\int t\|z'\|^3 \quad .$$

We have to find bounds of the terms on the right hand side separately. The first is bounded by - see Lemma 1

$$\tfrac{2}{3}t\ z^3(1) \le c_3 t\|z\|^{3/2}\|z'\|^{3/2}$$

(4.16) $$\le \tfrac{1}{2}t\|z'\|^2 + c_4 t\|z\|^6$$

$$\le \tfrac{1}{2}t\|z'\|^2 + k_1 \quad .$$

The second term is already bounded via Lemma 1. Finally we have

(4.17) $$\left|\int z^3(1)\right| \le c_5 \int\|z\|^{3/2}\|z'\|^{3/2} \le k_2$$

because of Lemma 1. In this way we come to the integral inequality

(4.18) $$t\|z'\|^2 + \int t\|\dot z\|^2 \le k_3 + k_4 \int t\|z'\|^3 \quad .$$

Similar to (4.7), (4.8) the solution λ of the integral equation

(4.19) $$\lambda = k_3 + k_4 \int t^{-1/2}\ \lambda^{3/2}$$

is a bound for the left hand side of (4.18). The solution is

(4.20) $$\lambda = k_3(1-k_4\sqrt{k_3 t})^{-2} \quad .$$

The choice

(4.21) $$T = \mathrm{Min}(T_1, 1/(4k_3k_4^2))$$

gives (4.11) with $k = 4k_3$.

Step 3: By differentiation of (1.14) ν-times with respect to t we get the defining relation for z_ν

(4.22) $(\dot z_\nu, \chi) + (z'_\nu, \chi') = \Sigma\binom{\nu}{\mu}\, z_\mu(1)(xz_{\nu-\mu}, \chi')$.

Here and in the following Σ means

(4.23) $$\Sigma(.) = \sum_{\mu=0}^{\nu}(.) \quad .$$

We will suppress the dependence on ν since there will be no confusion. We will also use Σ' with the meaning

(4.24) $$\Sigma'(.) = \sum_{\mu=1}^{\nu-1}(.) \quad .$$

In the same way Σ, Σ' will denote certain sums, the different sums entering the formulae will be $\Sigma_1, \Sigma'_1, \Sigma_2$ etc. .

The choice $\chi = z_\nu$ in (4.22) gives

(4.25) $\frac{1}{2}\partial_t\|z_\nu\|^2 + \|z'_\nu\|^2 = \Sigma\binom{\nu}{\mu}\, z_\mu(1)(xz_{\nu-\mu}, z'_\nu) =: \Sigma_1$.

The indices $\mu = 0$ and $\mu = \nu$ play a special role. It is

(4.26) $\Sigma_1 = z_o(1)(xz_\nu, z'_\nu) + z_\nu(1)(xz_o, z'_\nu) + \Sigma'_1$.

With the help of Lemmata 1 and 2 we come to

(4.27) $$|\Sigma_1| \le k_5 t^{-1/4}\|z_\nu\|\|z'_\nu\| + k_6\|z_\nu\|^{1/2}\|z'_\nu\|^{3/2} + |\Sigma'_1| \le$$
$$\le \tfrac{1}{4}\|z'_\nu\|^2 + k_7 t^{-1/2}\|z_\nu\|^2 + |\Sigma'_1| \quad .$$

The sum Σ'_1 still depends on z'_ν but we can estimate

(4.28) $$|\Sigma'_1| \le \|z'_\nu\|\left\{\Sigma'\binom{\nu}{\mu}\ |z_\mu|\|z_{\nu-\mu}\|\right\} =: \|z'_\nu\|\Sigma'_2$$
$$\le \tfrac{1}{4}\|z'_\nu\|^2 + (\Sigma'_2)^2 \quad .$$

With (4.27) and (4.28) we get from (4.25)

(4.29) $\partial_t\|z_\nu\|^2 + \|z'_\nu\|^2 \le 2k_7 t^{-1/2}\|z_\nu\|^2 + 2(\Sigma'_2)^2$.

By our induction hypothesis we have

(4.30) $$\Sigma'_2 \le t^{-\nu-1/4}\,\rho_\nu$$

with

(4.31) $$\rho_\nu = \Sigma'\binom{\nu}{\mu}\ \bar\gamma_\mu\,\bar\gamma_{\nu-\mu} \quad .$$

Multiplication of (4.29) with $t^{2\nu}$ and integration leads to

(4.32) $$t^{2\nu}\|z_\nu\|^2 + \int t^{2\nu}\|z'_\nu\|^2 \le (2\nu+k_8)\int t^{2\nu-1}\|z_\nu\|^2 + {}$$
$$+ 2\rho_\nu^2\int t^{-1/2}$$
$$\le (2\nu+k_8)\ \gamma_{2\nu-1}^2 + \rho_\nu^2\, T^{1/2} \quad .$$

Since the first term is covered by the second we have

<u>LEMMA 3:</u> <u>Let</u> $\bar\gamma_\mu$ <u>for</u> $\mu \le \nu - 1$ <u>be known</u>. <u>Then</u>

(4.33) $$\gamma_{2\nu} \le \varkappa' \sum_{1}^{\nu-1}\binom{\nu}{\mu}\ \bar\gamma_\mu\,\bar\gamma_{\nu-\mu}$$

<u>with</u> $\varkappa'$ <u>depending only on</u> $\|g\|$.

<u>Step 4:</u> Now we take $\chi = \dot z_\nu = z_{\nu+1}$ in (4.22) giving

(4.34) $\|z_{\nu+1}\|^2 + \frac{1}{2}\partial_t\|z'_\nu\|^2 = \Sigma\binom{\nu}{\mu}z_\mu(1)(xz_{\nu-\mu}, \dot z'_\nu) =: \Sigma_3$.

Similar to Steps 2 and 3 we have to integrate by parts and have to consider the indices $\mu = 0$ and $\mu = \nu$ separately. It is

$$\begin{aligned}\Sigma_3 &= \Sigma_3' + z_o(1)(xz_\nu, \dot z_\nu') + z_\nu(1)(xz_o, \dot z_\nu') \\ &= \Sigma_3' + 2z_o(1)\, z_\nu(1)\, \dot z_\nu(1) - z_o(1)(xz_\nu' + z_\nu, \dot z_\nu) - \\ &\quad - z_\nu(1)(xz_o' + z_o, z_\nu) \\ &=: \Sigma_3' + A_2 + A_3 + A_4 \quad .\end{aligned} \tag{4.35}$$

Firstly we will analyze the last three terms. In estimating

$$A_2 = \partial_t(z_o(1) z_\nu^2(1)) - \dot z_o(1) z_\nu^2(1)) \tag{4.36}$$

we have to consider the cases $\nu = 1$ and $\nu \geq 2$ separately. In the latter case we have already bounded γ_2 and γ_3 . This leads to

$$|\dot z_o(1)| \leq \bar\gamma_1\ t^{-5/4} \quad . \tag{4.37}$$

Then we get

$$\begin{aligned}\int t^{2\nu+1} A_2 &\leq t^{2\nu+1} z_o(1) z_\nu^2(1) + \\ &\quad + (2\nu+1) \int t^{2\nu} z_o(1) z_\nu^2(1) + \\ &\quad + \bar\gamma_1 \int t^{2\nu-1/4} |z_\nu(1)|^2\end{aligned} \tag{4.38}$$

and because of Lemmata 1 and 2

$$\begin{aligned}\int t^{2\nu+1} A_2 &\leq k_9\ t^{2\nu+3/4} \|z_\nu\| \|z_\nu'\| + \\ &\quad + k_{10}\nu \int t^{2\nu-1/4} \|z_\nu\| \|z_\nu'\| \quad .\end{aligned} \tag{4.39}$$

With $\delta > 0$ we get

$$\int t^{2\nu+1} A_2 \leq \delta\ t^{2\nu+1} \|z_\nu'\|^2 + k_{11}(1+\tfrac{1}{\delta})(\gamma_{2\nu}^2 + \nu^2 \bar\gamma_{\nu-1}^2) \quad . \tag{4.40}$$

In the excluded case $\nu = 1$ the second term in (4.36) is $z_1^3(1)$ and we get by applying Young's inequality

$$\begin{aligned}\int t^3 |z_1|^3 &\leq 2\sqrt{2} \int t^3 \|z_1\|^{3/2} \|z_1'\|^{3/2} \\ &\leq c_6 \left(\int t^2 \|z_1'\|^2 + \int t^6 \|z_1\|^6\right) \\ &\leq c_6(\gamma_2^2 + T\gamma_2^6) \quad .\end{aligned} \tag{4.41}$$

Therefore (4.40) also is valid for $\nu = 1$ (possibly with k_{11} changed).

Next we have with $\delta > 0$ arbitrary

$$\begin{aligned}A_3 &\leq 2\bar\gamma_o\ t^{-1/4} \|z_\nu'\| \|\dot z_\nu\| \\ &\leq \delta \|\dot z_\nu\|^2 + \delta^{-1} k_{12}\ t^{-1/2} \|z_\nu'\|^2\end{aligned} \tag{4.42}$$

This gives

$$\begin{aligned}\int t^{2\nu+1} A_3 &\leq \delta \int t^{2\nu+1} \|z_{\nu+1}\|^2 \\ &\quad + \delta^{-1} k_{12} \int t^{2\nu+1/2} \|z_\nu'\|^2 \\ &\leq \delta \int t^{2\nu+1} \|z_{\nu+1}\|^2 + \delta^{-1} k_{13} \gamma_{2\nu}^2 \quad .\end{aligned} \tag{4.43}$$

The estimation of A_4 follows the same lines and leads to a bound of the above type. Thus we have

$$\begin{aligned}\int t^{2\nu+1}(A_2 + A_3 + A_4) &\leq 2\delta\left\{t^{2\nu+1} \|z_\nu'\|^2 + \int t^{2\nu+1} \|z_{\nu+1}\|^2\right\} + \\ &\quad + k_{14}(1+\delta^{-1})(\gamma_{2\nu}^2 + \nu^2\ \bar\gamma_{\nu-1}^2) \quad .\end{aligned} \tag{4.44}$$

It remains to analyze Σ_3' . By partial integration we get

$$\Sigma_3' = \Sigma'\binom{\nu}{\mu} z_\mu(1)\, z_{\nu-\mu}(1)\, \dot z_\nu(1) - \tag{4.45}$$
$$- \Sigma'\binom{\nu}{\mu} z_\mu(1)(xz'_{\nu-\mu} + z_{\nu-\mu}, \dot z_\nu)$$
$$=: \Sigma_4' + \Sigma_5' \quad .$$

To derive a bound for Σ_5' is short

$$\Sigma_5' \le 2\|\dot z_\nu\|\ \Sigma'\binom{\nu}{\mu}\ |z_\mu|\,\|z'_{\nu-\mu}\| \tag{4.46}$$
$$\le \delta\|\dot z_\nu\|^2 + \delta^{-1}\ t^{-2\nu-3/2}\ \rho_\nu^2$$

leading to

$$\int t^{2\nu+1}\ \Sigma_5' \le \delta \int t^{2\nu+1}\|\dot z_\nu\|^2 + 2\delta^{-1}\ T^{1/2}\ \rho_\nu^2 \quad . \tag{4.47}$$

The estimation of Σ_4' is more lengthy. We may write

$$\Sigma_4' = \partial_t\Big\{z_\nu(1)\, \Sigma'\binom{\nu}{\mu}\ z_\mu(1)\ z_{\nu-\mu}(1)\Big\} - \tag{4.48}$$
$$- 2z_\nu(1)\ \Sigma'\binom{\nu}{\mu}\ \dot z_\mu(1)\ z_{\nu-\mu}(1)$$
$$=: \partial_t\big\{z_\nu(1)\ \Sigma_6'\big\} + 2z_\nu(1)\ \Sigma_7' \quad .$$

We have

$$\int t^{2\nu+1}\ \partial_t\big\{z_\nu(1)\ \Sigma_6'\big\} = t^{2\nu+1}\ z_\nu(1)\ \Sigma_6' - \tag{4.49}$$
$$- (2\nu+1) \int t^{2\nu}\ z_\nu(1)\ \Sigma_6' \quad .$$

Let us consider the second term firstly. With the estimates already won - or assumed - for $\mu \le \nu-1$ we get

$$|t^{2\nu} z_\nu(1)\ \Sigma_6'| \le \sqrt{2}\ t^{\nu-1/2}\rho_\nu\|z_\nu\|^{1/2}\|z_\nu'\|^{1/2} \tag{4.50}$$
$$\le \sqrt{2}(t^{2\nu-1}\|z_\nu\|^2)^{1/4}(t^{2\nu}\|z_\nu'\|^2)^{1/4}(t^{-1/4}\rho_\nu) \quad .$$

Now we apply the inequality

$$a^{1/4}\ b^{1/4}\ c \le \frac{1}{4}\ a + \frac{1}{4}\ b + \frac{1}{2}\ c^2 \quad . \tag{4.51}$$

The additional factor $2\nu+1$ is taken to the first term. This gives

$$(2\nu+1)\Big|\int t^{2\nu} z_\nu(1)\ \Sigma_6'\Big| \le c_7\Big\{\nu^{4}\gamma_{\nu-1}^{2} + \gamma_{2\nu}^2 + T^{1/2}\rho_\nu^2\Big\} \quad . \tag{4.52}$$

In the analogue way we find

$$t^{2\nu+1} z_\nu(1)\ \Sigma_6' \le \tag{4.53}$$
$$\le \delta\ t^{2\nu+1}\|z_\nu'\|^2 + c_8(1+\delta^{-1})\Big\{\gamma_{2\nu}^2 + T^{1/2}\rho_\nu^2\Big\} \quad .$$

Finally we turn over to Σ_7'. Because of $\dot z_\mu = z_{\mu+1}$ we get

$$\Sigma_7' = \nu\ z_1(1)\ z_\nu(1) + \sum_1^{\nu-2}\binom{\nu}{\mu} z_{\mu+1}(1) z_{\nu-\mu}(1) \quad . \tag{4.54}$$

Once more we have to differentiate between $\nu = 1$ and $\nu > 1$. In the latter case we have already a bound for $z_1(1)$ leading to

$$2|z_\nu(1)\Sigma_7'| \le k_{15}\nu\ t^{-5/4}\|z_\nu\|\|z_\nu'\| + \tag{4.55}$$
$$+ c_9\|z_\nu\|^{1/2}\|z_\nu'\|^{1/2} \cdot t^{-\nu-3/2}\ \tilde\rho_\nu$$

with

$$\tilde\rho_\nu = \sum_1^{\nu-2}\binom{\nu}{\mu}\bar\gamma_{\mu+1}\ \bar\gamma_{\nu-\mu} \tag{4.56}$$

what can be further estimated by - see (4.51)

$$(4.57)\quad 2t^{2\nu+1}|z_\nu(1)\ \Sigma_7'| \le$$
$$\le c_{10}\left\{t^{2\nu}\|z_\nu'\|^2 + \nu^2 t^{2\nu-1}\|z_\nu\|^2 + t^{-1/2}\ \tilde{\rho}_\nu^2\right\}$$

giving

$$(4.58)\quad \left|2\int t^{2\nu+1} z_\nu(1)\ \Sigma_7'\right| \le 2c_{10}(\gamma_{2\nu}^2+\nu^2\ \overline{\gamma}_{\nu-1}^2+T^{1/2}\ \tilde{\rho}_\nu^2) \quad .$$

The case $\nu = 1$ gives

$$(4.59)\quad \Sigma_7' = \emptyset$$

and nothing has to be proved.

Now we are ready to put things together. With the help of (4.44), (4.47), (4.52), (4.53), and (4.58) we derive from (4.34)

$$(4.60)\quad t^{2\nu+1}\|z_\nu'\|^2 + \int t^{2\nu+1}\|z_{\nu+1}\|^2 \le$$
$$\le 3\delta(t^{2\nu+1}\|z_\nu'\|^2 + \int t^{2\nu+1}\|z_{\nu+1}\|^2) +$$
$$+ (2\nu+1)\int t^{2\nu}\|z_\nu'\|^2 +$$
$$+ k_{16}(1+\delta^{-1})(\gamma_{2\nu}^2+\nu^4\ \overline{\gamma}_{\nu-1}^2+\rho_\nu^2+\tilde{\rho}_\nu^2) \quad .$$

With $\delta < 1/6$ we get

<u>LEMMA 4:</u> <u>Let</u> $\overline{\gamma}_\mu$ <u>for</u> $\mu \le \nu-1$ <u>and</u> $\gamma_{2\nu}$ <u>be known. Then</u>

$$\gamma_{2\nu+1} \le \varkappa''\left\{\sqrt{\nu}\ \gamma_{2\nu} + \nu^2\ \overline{\gamma}_{\nu-1} + \rho_\nu + \tilde{\rho}_\nu\right\}$$

<u>with</u> $\varkappa''$ <u>depending only on</u> $\|g\|$.

By means of Lemmata 3 and 4 we have the recurrence relation

$$(4.61)\quad \overline{\gamma}_\nu \le \tilde{\varkappa}\left\{\sqrt{\nu}\ \sum_1^{\nu-1}\binom{\nu}{\mu}\overline{\gamma}_\mu\overline{\gamma}_{\nu-\mu} + \sum_1^{\nu-2}\binom{\nu}{\mu}\overline{\gamma}_{\mu+1}\overline{\gamma}_{\nu-\mu} + \nu^2\ \overline{\gamma}_{\nu-1}\right\}$$

with $\tilde{\varkappa}$ depending only on $\|g\|$. The term with $\mu = \nu-2$ in the second sum is covered by the last term. In this way we get

$$(4.62)\quad \overline{\gamma}_\nu \le \varkappa\left\{\sqrt{\nu}\ \sum_1^{\nu-1} \dots + \sum_1^{\nu-3} \dots + \nu^2\ \overline{\gamma}_{\nu-1}\right\}$$

with a new $\varkappa = \varkappa(\|g\|)$.

Without loss of generality we can assume $\overline{\gamma}_1 \le \varkappa$, otherwise $\varkappa$ has to be increased. Our aim is to bound $\overline{\gamma}_\mu$ by

$$(4.63)\quad \Gamma_\mu := (\mu!)^2\ \varkappa(\vartheta\varkappa)^{2(\mu-1)}$$

with some ϑ fixed depending only on $\|g\|$. Then the remaining inequality (3.3) of the theorem is proven.

Of course for any $N = N(\|g\|)$ fixed there is a ϑ such that

$$(4.64)\quad \overline{\gamma}_\mu \le \Gamma_\mu \qquad \text{for } \mu = 1,2,\dots,N \quad .$$

The integer $N = N(\|g\|)$ will be specified later on, see (4.74). Now we apply complete induction: Let $\nu > N$. We assume $\overline{\gamma}_\mu \le \Gamma_\mu$ for $\mu \le \nu-1$ already proven. Then (4.62) gives

$$\bar{\gamma}_\nu \le \varkappa\Big\{\sqrt{\nu}\sum_{1}^{\nu-1}\binom{\nu}{\mu}(\mu!)^2((\nu-\mu)!)^2\vartheta^{2\nu-4}\varkappa^{2\nu-2} + \sum_{1}^{\nu-3}\binom{\nu}{\mu}((\mu+1)!)^2((\nu-\mu)!)^2\vartheta^{2\nu-2}\varkappa^{2\nu} + (\nu!)^2\vartheta^{2\nu-4}\varkappa^{2\nu-3}\Big\} \quad . \tag{4.65}$$

With the abbreviations

$$\begin{aligned}\Sigma^1 &= \sqrt{\nu}(\nu!)^{-2}\sum_{1}^{\nu-1}\binom{\nu}{\mu}(\mu!)^2((\nu-\mu)!)^2 \quad ,\\ \Sigma^2 &= (\nu!)^{-2}\sum_{1}^{\nu-3}\binom{\nu}{\mu}((\mu+1)!)^2((\nu-\mu)!)^2\end{aligned} \tag{4.66}$$

we can rewrite (4.65) in the form

$$\bar{\gamma}_\nu \le \Big\{(\nu!)^2\varkappa(\vartheta\varkappa)^{2(\nu-1)}\Big\}\Big\{\vartheta^{-2}\Sigma^1 + \varkappa^2\Sigma^2 + \vartheta^{-2}\varkappa^{-1}\Big\} \quad . \tag{4.67}$$

We need bounds of the sums Σ^1 , Σ^2 . Since only $\nu \to \infty$ is of interest we may assume $\nu \ge 9$. Then

$$\begin{aligned}\binom{\nu}{\mu} = \binom{\nu}{\nu-\mu} &\ge c_{10}\,\nu^\mu \quad \text{for} \quad \mu = 1,2,3,\\ \binom{\nu}{\mu} &\ge c_{10}\,\nu^4 \quad \text{for} \quad 4 \le \mu \le \nu-4 \quad .\end{aligned} \tag{4.68}$$

It is

$$\Sigma^1 = \sqrt{\nu}\sum_{1}^{\nu-1}\binom{\nu}{\mu}^{-1} \tag{4.69}$$

and therefore

$$\begin{aligned}\Sigma^1 &\le 2\sqrt{\nu}\Big\{\nu^{-1} + c_{10}\sum_{2}^{\nu/2}\nu^{-2}\Big\}\\ &\le c_{11}\,\nu^{-1/2} \quad .\end{aligned} \tag{4.70}$$

Similarily we have

$$\begin{aligned}\Sigma^2 &= \sum_{1}^{\nu-3}(\nu!)^{-1}(\mu+1)((\mu+1)!)((\nu-\mu)!)\\ &\le 4\sum_{1}^{\nu-3}\mu^2\binom{\nu}{\mu}^{-1} \quad .\end{aligned} \tag{4.71}$$

By means of (4.68) we can estimate

$$\begin{aligned}\Sigma^2 &\le c_{12}\Big\{\frac{1}{\nu} + \sum_{4}^{\nu-4}\mu^2\binom{\nu}{\mu}^{-1}\Big\}\\ &\le c_{13}\,\nu^{-1} \quad .\end{aligned} \tag{4.72}$$

In this way we get from (4.67)

$$\bar{\gamma}_\nu \le \Big\{(\nu!)^2\varkappa(\vartheta\varkappa)^{2(\nu-1)}\Big\}\Big\{c_{11}\vartheta^{-2} + c_{13}\varkappa^2\nu^{-1} + \vartheta^{-2}\varkappa^{-1}\Big\}. \tag{4.73}$$

Now the choice of ϑ and N is obvious. We take

$$\begin{aligned}N &= \Big[1 + 2\,c_{13}\,\varkappa^2\Big]\\ \vartheta \ge \vartheta_0 &= \Big\{2(c_{11} + \varkappa^{-1})\Big\}^{1/2}\end{aligned} \tag{4.74}$$

what depends only on $\|g\|$. Then the second brackets in (4.73) are less than 1 for $\nu > N$, i.e. we have $\bar{\gamma}_\nu < \Gamma_\nu$ by induction.

Besides (4.74_2) ϑ has to be chosen such that $\bar{\gamma}_\mu \le \Gamma_\mu$ for $\mu \le N$. But also this is only dependent on $\|g\|$.

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