To appear in Proceedings of a Special Year in Numerical Analysis, University of Maryland

# SCHAUDER ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS ON SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

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## 0. Introduction

Let $u$ be the solution of a second order elliptic boundary value problem and let $u_h = R_h u \in S_h$ be the corresponding Ritz resp. finite element approximation onto the space $S_h$. Asking for $L_\infty$-estimates of $u_h$ itself or the error $u-u_h$ for approximation spaces $S_h$ of order at least $3$, i.e. for finite elements which are at least piecewise quadratics, the following results are to be mentioned:

(i) In Scott [9] for $N = 2$ dimensions it is proven

$$(0.1) \qquad \|u-R_h u\|_{L_\infty} \le c\ h \inf_{\chi\in S_h} \|\nabla(u-\chi)\|_{L_\infty} .$$

The proof is based on a careful analysis of the approximability of the Green's function in the norm of $W_1^1$.

(ii) In Nitsche [6] for arbitrary dimensions the a priori estimate

$$(0.2) \qquad \|R_h u\|_{L_\infty} + h\|\nabla(R_h u)\|_{L_\infty} \le c\left\{\|u\|_{L_\infty} + h\|\nabla u\|_{L_\infty}\right\}$$

was shown. Generalyzing earlier results of Natterer the proof is based on the extensive use of certain weighted norms which are in the case of finite elements strongly connected with $L_\infty$-norms.

(iii) In Schatz-Wahlbin [8] the estimate

$$(0.3) \qquad \|R_h u\|_{L_\infty} \le c\|u\|_{L_\infty}$$

is proven. The method used is somehow between the other two mentioned above.

The first aim of the present paper is to show that the estimate (0.3) can be derived directly following the lines of our former paper with the only difference that whenever the gradient of $u$ enters the formulae then partial integration has to be applied. Actually this happens only in three places. In order to give a self-contained representation we repeat the arguments of our paper, the only changes are explained in Remarks 5 and 6. For the sake of simplicity resp. clearness we give the analysis in Section 3 for the Laplacian serving as a model problem. The case of a general second order equation causes no additional difficulties, this is discussed in Section 6. The proof of a crucial lemma was skipped in our former paper. It is given in detail in Section 4.

The second aim of this paper is to show the boundedness of the Ritz operator in Hoelder- resp. Lipschitz spaces. These spaces are the adequate ones in treating nonlinear elliptic problems. The boundedness of the Ritz operator in the corresponding norms at least simplifies the analysis of finite element procedures, in some cases it is essential.

Seemingly up to now Hoelder spaces did not find any attention in the finite element literature. Corresponding to this a priori estimates or error estimates in the norms of these spaces do not exist in the literature.



## 1. Notations, Finite Elements

In the following $\Omega \subseteq \mathbb{R}^N$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. For any $\Omega' \subseteq \Omega$ let $W_p^k(\Omega')$ be the Sobolev space of functions having $L_p$-integrable derivatives of order up to $k$. The norms are indicated by the corresponding subscripts. In the case $p = 2$ we also adopt $H_k(\Omega') = W_2^k(\Omega')$. The norms then are written shortly

(1.1) $$\|\cdot\|_{k.\Omega'} = \|\cdot\|_{W_2^k(\Omega')} .$$

In addition we will use the abbreviation for boundary norms:

(1.2) $$|\cdot|_{k.\Omega'} = \|\cdot\|_{W_2^k(\partial\Omega')} .$$

Moreover, $\Omega'$ is skipped in case of $\Omega' = \Omega$ and $k$ in case of $k = 0$.

The use of weighted norms resp. semi-norms will be essential. They are defined by

(1.3) $$\|\nabla^k v\|_{\alpha.\Omega'} = \Big\{ \sum_{|\xi|=k} \iint_{\Omega'} \mu^{-\alpha} |D^\xi v|^2 dx \Big\}^{1/2}$$

with $\mu$ given by

(1.4) $$\mu = \mu(x) = |x - x_o|^2 + \rho^2$$

($x_o \in \bar{\Omega}$, $\rho > 0$). The boundary semi-norms $|\cdot|_{\alpha.\Omega'}$ are defined in the corresponding way.

By $\Gamma_h$ a subdivision of $\Omega$ into generalized simplices $\Delta$ is meant, i.e. $\Delta$ is a simplex if $\Delta$ intersects $\partial\Omega$

in at most a finite number of points and otherwise one of the faces may be curved. $\Gamma_h$ is called $\varkappa$-regular if to any $\Delta \in \Gamma_h$ there are two spheres of diameters $\varkappa^{-1}h$ and $h$ such that $\Delta$ contains the one and is contained in the other.

The finite element spaces $S_h = S(\Gamma_h)$ we will work with have the following structure: Let $m$ be an integer fixed. Any element of $S_h$ is continuous in $\Omega$ and the restriction to $\Delta \in \Gamma_h$ is a polynomial of degree less than $m$. In curved elements we use isoparametric modifications as discussed by CIARLET-RAVIART [2], ZLAMAL [10]. $\overset{o}{S}_h$ is the intersection of $S_h$ and $\overset{o}{H}_1$, the closure in $H_1$ of the functions with compact support.

By construction we have $S_h \subseteq H_1$ but in general $S_h \nsubseteq H_k$ for $k \geq 2$. It is useful to introduce the spaces $H_k' = H_k'(\Gamma_h)$ consisting of functions the restriction of which to any $\Delta$ is in $H_k(\Delta)$. Obviously $S_h \subseteq H_k'$ for all $k$.

Parallel to above we use 'broken' seminorms

(1.5)
$$\|\nabla^k v\|_\alpha' = \left\{ \sum_{\Delta \in \Gamma_h} \|\nabla^k v\|_{\alpha,\Delta}^2 \right\}^{1/2} ,$$
$$|\nabla^k v|_\alpha' = \left\{ \sum_{\Delta \in \Gamma_h} |\nabla^k v|_{\alpha,\Delta}^2 \right\}^{1/2} .$$



## 2. Approximation Theory in Weighted Norms

In the estimates of the next sections $c$, $c_1$ etc. will denote generic constants which may differ at different locations. Unless otherwise stated they depend only on (i) the domain $\Omega$, (ii) the dimension $N$, (iii) the regularity parameter $\varkappa$, and (iv) the order $m$.

Essential is the fact that the function $\mu$ (1.4) does not change too fast in any $\Delta \in \Gamma_h$ if $\rho$ is not small compared with $h$:

__Lemma 1:__ __Let__ $\rho \geq h$. __Then for any__ $\Delta \in \Gamma_h$

(2.1)
$$\bar{\mu}_\Delta = \sup_{x \in \Delta} \mu(x) \leq 6 \inf_{x \in \Delta} \mu(x) = 6\underline{\mu}_\Delta .$$

__Proof:__ Let $\bar{x}, \underline{x} \in \bar{\Delta}$ be points where $\mu$ attains its maximum and minimum. Then

(2.2)
$$\bar{\mu}_\Delta = \mu(\bar{x}) = \mu(\underline{x}) + (\bar{x}-\underline{x}) \cdot \nabla\mu(\tilde{x}) .$$

Now we have

(2.3)
$$|\nabla\mu(\tilde{x})| = 2|\tilde{x}-x_o| \leq 2\bar{\mu}_\Delta^{1/2}$$

and

(2.4)
$$|\bar{x}-\underline{x}| \leq h \leq \rho \leq \underline{\mu}_\Delta^{1/2}$$

leading to

(2.5)
$$\bar{\mu}_\Delta \leq \underline{\mu}_\Delta + 2\underline{\mu}_\Delta^{1/2}\,\bar{\mu}_\Delta^{1/2}$$
$$\leq 3\underline{\mu}_\Delta + \frac{1}{2}\bar{\mu}_\Delta . \qquad \#$$

Next let $v \in C^o \cap H_1'$ be given and $\chi \in S_h$ an appropriate interpolation. Then the estimate

$$(2.6) \qquad \|\nabla^k(v-\chi)\|^2_{L_2(\Delta)} \le c\, h^{2(1-k)} \|\nabla^1 v\|^2_{L_2(\Delta)}$$

for any $\Delta \in \Gamma_h$ and $0 \le k < 1 \le m$ is well known. Because of Lemma 1 we derive from this

$$(2.7) \qquad \|\nabla^k(v-\chi)\|^2_{\alpha.\Delta} \le c\, 6^{|\alpha|}\, h^{2(1-k)} \|\nabla^1 v\|^2_{\alpha.\Delta} \quad .$$

The power $\alpha$ will be within the range $|\alpha| \le N+1$ . Thus we drop the factor $6^{|\alpha|}$. Summation over all $\Delta \in \Gamma_h$ gives

<u>Lemma 2</u>: <u>Let</u> $\rho \ge h$ . <u>To any</u> $v \in C^o \cap H_1'$ <u>there is a</u> $\chi \in S_h$ <u>according to</u>

$$(2.8) \qquad \|\nabla^k(v-\chi)\|^o_\alpha \le c\, h^{1-k} \|\nabla^1 v\|^o_\alpha \quad ,$$

<u>for</u> $0 \le k < 1 \le m$ .

<u>Remark 1</u>: Since (2.7) is valid also for $v \in C^o \cap H_1' \cap \overset{o}{H}_1$ with $\chi \in \overset{o}{S}_h$ the lemma remains valid in this situation.

For any $w \in H_1(\Delta)$ the trace theorem gives

$$(2.9) \qquad \|w\|^2_{L_2(\partial\Delta)} \le c\left\{h^{-1}\|w\|^2_{L_2(\Delta)} + h\|\nabla w\|^2_{L_2(\Delta)}\right\} \quad .$$

Using the arguments of above we get

<u>Corollary 2</u>: <u>Under the assumptions of Lemma 2</u>

$$(2.10) \qquad |\nabla^k(v-\chi)|'_\alpha \le c\, h^{1-k-\frac{1}{2}} \|\nabla^1 v\|'_\alpha$$

<u>is valid in addition</u>.

The proof of the next lemma and corollary follows the same lines and is omitted here.



<u>Lemma 3</u>: <u>For</u> $\chi \in S_h$ <u>and</u> $0 \le k < 1 < m$ <u>inverse relations of the type</u>

$$(2.11) \qquad \|\nabla^1 \chi\|'_\alpha \le c\, h^{-(1-k)} \|\nabla^k \chi\|'_\alpha$$

<u>hold true</u>.

<u>Corollary 3</u>: <u>In addition to</u> (2.11)

$$(2.12) \qquad |\nabla^1 \chi|'_\alpha \le c\, h^{-(1-k+\frac{1}{2})} \|\nabla^k \chi\|'_\alpha$$

<u>holds true</u>. <u>Here</u> $k = 1$ <u>is accepted</u>.

In the subsequent sections we will apply these approximation results to functions $v$ of the structure $v = \mu^{-\alpha}\varphi$ with $\varphi \in S_h$ . Then a certain super-approximability property holds:

<u>Lemma 4</u>: <u>Let</u> $\varphi \in S_h$ <u>be given</u>. <u>The function</u> $\mu^{-\alpha}\varphi$ <u>can be approximated by an element</u> $\chi \in S_h$ <u>according to</u>

$$(2.13) \qquad h\|\nabla^2(\mu^{-\alpha}\varphi-\chi)\|'_{-\alpha} + \|\nabla(\mu^{-\alpha}\varphi-\chi)\|_{-\alpha} + h^{1/2}|\nabla(\mu^{-\alpha}\varphi-\chi)|'_{-\alpha} \le c\frac{h}{\rho}(\|\varphi\|_{\alpha+1} + \|\nabla\varphi\|_\alpha) \quad .$$

<u>Proof</u>: We apply Lemma 2 and Corollary 2 with $1 = m$ and get the bound

$$(2.14) \qquad c\, h^{m-1}\|\nabla^m(\mu^{-\alpha}\varphi)\|'_{-\alpha}$$

for the three terms on the left hand side in (2.13). Since $\varphi$ is piecewise a polynomial of degree less than $m$ and because of

$$(2.15) \qquad |D^\xi \mu^{-\alpha}| \le c\, \mu^{-\alpha-|\xi|/2}$$

Leibniz' rule gives

$$(2.16) \qquad \|\nabla^m(\mu^{-\alpha}\varphi)\|'_{-\alpha} \le c \sum_{n=0}^{m-1} \|\nabla^n\varphi\|'_{\alpha+m-n} \quad .$$

Now we apply Lemma 3 for the terms with $n \ge 1$ :

$$(2.17) \qquad \|\nabla^m(\mu^{-\alpha}\varphi)\|'_{-\alpha} \le c\left\{\|\varphi\|_{\alpha+m} + \sum_{1}^{m-1} h^{1-n}\|\nabla\varphi\|_{\alpha+m-n}\right\} \quad .$$

Using finally the obvious inequality for $\beta > 0$

$$(2.18) \qquad \|\cdot\|_{(\cdot)+\beta} \le \|\cdot\|_{(\cdot)}\rho^{-\beta}$$

we end up with

$$(2.19) \qquad \|\nabla^m(\mu^{-\alpha}\varphi)\|'_{-\alpha} \le c\left\{\rho^{1-m}\|\varphi\|_{\alpha+1} + \sum_{1}^{m-1} h^{1-n}\,\rho^{n-m}\|\nabla\varphi\|_{\alpha}\right\}$$

and therefore

$$(2.20) \qquad c\, h^{m-1}\|\nabla^m(\mu^{-\alpha}\varphi)\|'_{-\alpha} \le$$

$$\le c\left\{(h/\rho)^{m-1} + \sum_{1}^{m-1}(h/\rho)^{m-n}\right\}\left\{\|\varphi\|_{\alpha+1} + \|\nabla\varphi\|_{\alpha}\right\} \quad .$$

The first brackets on the right hand side are bounded by $mh/\rho$ since $h \le \rho$ is assumed. #

As was pointed out in the introduction weighted norms are strongly connected with the $L_\infty$-norm. First we show

<u>Lemma 5</u>: <u>Let</u> $\alpha > \frac{N}{2}$ . <u>Then for any</u> $v \in L_\infty$ <u>it is</u>

$$(2.21) \qquad \|v\|_\alpha^2 \le c\, \rho^{-2\alpha+N}\|v\|_{L_\infty}^2 \quad .$$

<u>Proof</u>: We can estimate

$$(2.22) \qquad \|v\|_\alpha^2 \le \|v\|_{L_\infty}^2 \iint_\Omega \mu^{-\alpha}\, dx$$

and further with $r$ denoting the distance $|x-x_o|$



$$(2.23) \qquad \iint_\Omega \mu^{-\alpha}\, dx \le c\int_0^\infty (r^2+\rho^2)^{-\alpha}\, r^{N-1}\, dr$$

$$\le c\int_0^\infty (r+\rho)^{N-1-2\alpha}\, dr \quad . \qquad \#$$

For elements in the space $S_h$ there is the counterpart:

<u>Lemma 6</u>: <u>Let</u> $\alpha > \frac{N}{2}$ <u>and</u> $h \le \rho$ . <u>Then for</u> $\chi \in S_h$ <u>the inequality</u>

$$(2.24) \qquad \|\chi\|_{L_\infty}^2 \le c\, \rho^{2\alpha}\, h^{-N} \sup_{x_o\in\Omega} \|\chi\|_\alpha^2$$

<u>holds true</u>.

<u>Proof</u>: Let $x_o \in \bar\Omega$ be chosen such that

$$(2.25) \qquad \chi(x_o) = \pm\, \|\chi\|_{L_\infty}$$

and let $\Delta_o$ be one of the simplices with $x_o \in \bar\Delta_o$ .

$\chi$ restricted to $\Delta_o$ is a polynomial of finite degree, i.e. an element of a finite dimensional space. In this case any two norms are equivalent. Since $\Delta_o$ is of size $h$ there is a constant $c$ depending only on $\kappa$ , $N$ , and $m$ such that

$$(2.26) \qquad \|\chi\|_{L_\infty(\Delta_o)}^2 \le c\, h^{-N}\|\chi\|_{L_2(\Delta_o)}^2 \quad .$$

Because of the choice of $x_o$ it is for $x \in \Delta_o$

$$(2.27) \qquad \rho^2 \le \mu(x) \le \rho^2 + h^2 \le 2\rho^2 \quad .$$

Therefore we get further

$$(2.28)\quad \begin{aligned} \|\chi\|^2_{L_\infty(\Delta_0)} &\le c\, h^{-N} \rho^{2\alpha} \|\chi\|^2_{\alpha\cdot\Delta_0} \\ &\le c\, h^{-N} \rho^{2\alpha} \|\chi\|^2_{\alpha} \quad . \end{aligned}$$ #

Remark 2: The last two lemmata show that the $\alpha$-norm and the $L_\infty$-norm are equivalent in the spaces $S_h$ .



## 3. The Boundedness of the Ritz Projection

In this section we restrict ourselves to the model problem

$$(3.1)\quad \begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega , \\ u &= 0 \quad \text{on} \quad \partial\Omega . \end{aligned}$$

The weak formulation is:

Find $u \in \overset{\circ}{H}_1$ such that

$$(3.2)\quad D(u,v) = (f,v)$$

holds for all $v \in \overset{\circ}{H}_1$ .

Here $D(\ldots)$ denotes the Dirichlet integral

$$(3.3)\quad D(u,v) = (\nabla u, \nabla v) = \iint_\Omega \Sigma\, u_{|i}\, v_{|i}\, dx \quad .$$

The Ritz-approximation $\varphi = R_h u \in \overset{\circ}{S}_h$ is characterized by the relation

$$(3.4)\quad D(\varphi,\chi) = (f,\chi) \quad \text{for} \quad \chi \in \overset{\circ}{S}_h$$

or alternately by

$$(3.5)\quad D(\varphi,\chi) = D(u,\chi) \quad \text{for} \quad \chi \in \overset{\circ}{S}_h \quad .$$

Remark 3: Throughout this section the letter $\varphi$ denotes the Ritz-approximation on $u$ .

In the first step of our analysis we derive a bound for the gradient of $\varphi$ in a weighted norm. It is

$$\|\nabla\varphi\|_\alpha^2 = (\nabla\varphi, \mu^{-\alpha}\,\nabla\varphi) = D(\varphi,\mu^{-\alpha}\varphi) - (\nabla\varphi, \varphi\,\nabla\mu^{-\alpha}) = D(\varphi,\mu^{-\alpha}\varphi) + \frac{1}{2}\iint \varphi^2 \Delta\mu^{-\alpha} \quad . \tag{3.6}$$

Because of

$$\Delta\mu^{-\alpha} \le c\ \mu^{-\alpha-1} \tag{3.7}$$

we get

$$\|\nabla\varphi\|_\alpha^2 \le D(\varphi,\mu^{-\alpha}\varphi) + c\|\varphi\|_{\alpha+1}^2 \quad . \tag{3.8}$$

Next we use the identity

$$D(\varphi,\mu^{-\alpha}\varphi) = D(\varphi,\mu^{-\alpha}\varphi-\chi) - D(u,\mu^{-\alpha}\varphi-\chi) + D(u,\mu^{-\alpha}\varphi) \tag{3.9}$$

valid for any $\chi \in S_h$ because of (3.5) . By the aid of Schwarz' inequality in the form

$$|D(v,w)| \le \|\nabla v\|_\alpha\ \|\nabla w\|_{-\alpha} \tag{3.10}$$

and Lemma 4 we find for the first term on the right hand side of (3.9) with $\chi$ chosen appropriately

$$|D(\mu^{-\alpha}\varphi-\chi,\varphi)| \le c\ \frac{h}{\rho}\left\{\|\nabla\varphi\|_\alpha + \|\varphi\|_{\alpha+1}\right\}\|\nabla\varphi\|_\alpha \le c\ \frac{h}{\rho}\left\{\|\nabla\varphi\|_\alpha^2 + \|\varphi\|_{\alpha+1}^2\right\} \quad . \tag{3.11}$$

Our aim is to avoid any derivatives of $u$ in the estimates. Therefore we have to apply partial integration in order to handle the two other terms in (3.9). We get



$$D(u,\mu^{-\alpha}\varphi-\chi) = \sum_{\Delta\in\Gamma_h} \oint_{\partial\Delta} u(\mu^{-\alpha}\varphi-\chi)_n\ d\,O - \sum_{\Delta\in\Gamma_h} \iint_\Delta u\ \Delta(\mu^{-\alpha}\varphi-\chi)\ dx \tag{3.12}$$

which may be estimated by

$$|D(u,\mu^{-\alpha}\varphi-\chi\| \le |u|'_\alpha\ |\nabla(\mu^{-\alpha}\varphi-\chi)|'_{-\alpha} + \|u\|_\alpha\ \|\Delta(\mu^{-\alpha}\varphi-\chi)\|'_{-\alpha} \quad . \tag{3.13}$$

If $\chi$ is chosen according to Lemma 4 then

$$|D(u,\mu^{-\alpha}\varphi-\chi)| \le c\ \frac{h}{\rho}\left\{\|\nabla\varphi\|_\alpha + \|\varphi\|_{\alpha+1}\right\} \cdot \left\{h^{-1/2}|u|'_\alpha + h^{-1}\|u\|_\alpha\right\} \quad . \tag{3.14}$$

In order to shorten the formulae we introduce

$$N_\alpha(u) := \left\{h^{-2}\|u\|_\alpha^2 + h^{-1}|u|'^2_\alpha + \|u\|_{\alpha+1}^2\right\}^{1/2} \quad . \tag{3.15}$$

Then we come to - note $h \le \rho$ -

$$|D(u,\mu^{-\alpha}\varphi-\chi)| \le c\ \frac{h}{\rho}\left\{\|\nabla\varphi\|_\alpha^2 + \|\varphi\|_{\alpha+1}^2\right\} + c\ N_\alpha(u)^2 . \tag{3.16}$$

Following the same lines but this time using Lemma 3 and Corollary 3 we get

$$|D(u,\mu^{-\alpha}\varphi)| \le c\left\{\|\nabla\varphi\|_\alpha + \|\varphi\|_{\alpha+1}\right\} N_\alpha(u) \quad . \tag{3.17}$$

Schwarz' inequality in the form

$$|AB| \le \delta\ A^2 + \frac{1}{4\delta}\ B^2 \tag{3.18}$$

for $0 < \delta < 1$ leads to

$$(3.19) \qquad |D(u,\mu^{-\alpha}\varphi)| \le \delta\left\{\|\nabla\varphi\|_\alpha^2 + \|\varphi\|_{\alpha+1}^2\right\} + \frac{c}{\delta} N_\alpha(u)^2 \quad .$$

Now we combine (3.9), (3.11), (3.16) and (3.19) with (3.8). This gives

$$(3.20) \qquad \|\nabla\varphi\|_\alpha^2 \le (c_1 \frac{h}{\rho} + \delta)\, \|\nabla\varphi\|_\alpha^2 + c\|\varphi\|_{\alpha+1}^2 + \frac{c}{\delta} N_\alpha(u)^2 \quad .$$

We choose $\delta = 1/3$ and impose the condition on $\rho$

$$(3.21) \qquad \rho \ge \gamma_1 h \quad \text{with} \quad \gamma_1 = \max(1,3c_1) \quad .$$

Then we get

$$(3.22) \qquad \|\nabla\varphi\|_\alpha^2 \le c_2\|\varphi\|_{\alpha+1}^2 + c\, N_\alpha(u)^2 \quad .$$

Remark 4: In (3.20) we used for the constant in front of $\|\nabla\varphi\|_\alpha$ the numbering $c_1$ since this special constant appeared in the condition (3.21). Similarily the constant $c_2$ in front of $\|\varphi\|_{\alpha+1}$ appears in a further condition.

Remark 5: In the analysis given in [6] we did not use partial integration. There $\|\nabla u\|_\alpha$ enters instead of $N_\alpha(u)$ .

In the second step we introduce the auxiliary function $w$ defined by

$$(3.23) \qquad \begin{aligned} -\Delta w &= \mu^{-\alpha-1}\varphi \quad &&\text{in} \quad \Omega \quad , \\ w &= 0 &&\text{on} \quad \partial\Omega \quad . \end{aligned}$$

The reason is obvious since then



$$(3.24) \qquad \|\varphi\|_{\alpha+1}^2 = D(\varphi,w)$$

which may be rewritten with $\chi \in \overset{o}{S}_h$ arbitrary

$$(3.25) \qquad \|\varphi\|_{\alpha+1}^2 = D(\varphi,w-\chi) - D(u,w-\chi) + D(u,w) \quad .$$

Using the definition of $w$ we get at once for the last term on the right hand side

$$(3.26) \qquad \begin{aligned} D(u,w) &= (u,\varphi)_{\alpha+1} \\ &\le \delta\|\varphi\|_{\alpha+1}^2 + \frac{1}{4\delta}\|u\|_{\alpha+1}^2 \quad . \end{aligned}$$

Using (3.22) we get for the first term with $0 < \delta < 1$

$$(3.27) \qquad \begin{aligned} |D(\varphi,w-\chi)| &\le \|\nabla\varphi\|_\alpha \|\nabla(w-\chi)\|_{-\alpha} \\ &\le \delta\|\nabla\varphi\|_\alpha^2 + \frac{1}{4\delta}\|\nabla(w-\chi)\|_{-\alpha}^2 \\ &\le c_2\, \delta\|\varphi\|_{\alpha+1}^2 + c\, N_\alpha(u)^2 + \frac{1}{4\delta}\|\nabla(w-\chi)\|_{-\alpha}^2 . \end{aligned}$$

Finally, the middle term on the right hand side of (3.25) has to be treated by partial integration. Similar to above we come to

$$(3.28) \qquad \begin{aligned} |D(u,w-\chi)| &\le |u|_\alpha' \, |\nabla(w-\chi)|_{-\alpha}' + \|u\|_\alpha \, \|\nabla^2(w-\chi)\|_{-\alpha}' \\ &\le N_\alpha(u)^2 + h|\nabla(w-\chi)|_{-\alpha}'^2 + h^2\|\nabla^2(w-\chi)\|_{-\alpha}'^2 . \end{aligned}$$

By means of the last three estimates we derive from (3.25)

$$(3.29) \qquad \begin{aligned} \|\varphi\|_{\alpha+1}^2 \le\; & (1+c_2)\,\delta\|\varphi\|_{\alpha+1}^2 + \frac{c}{\delta} N_\alpha(u)^2 + \\ & + \frac{c}{\delta}\Big\{\|\nabla(w-\chi)\|_{-\alpha}^2 + h|\nabla(w-\chi)|_{-\alpha}'^2 + \\ & + h^2\|\nabla^2(w-\chi)\|_{-\alpha}'^2\Big\} \quad . \end{aligned}$$

The choice $\delta^{-1} = 2 + 2c_2$ leads to

$$(3.30)\qquad \|\varphi\|^2_{\alpha+1} \le c\, N_\alpha(u)^2 + c\Big\{\|\nabla(w-\chi)\|^2_{-\alpha} + h|\nabla(w-\chi)|'^2_{-\alpha} + h^2\|\nabla^2(w-\chi)\|'^2_{-\alpha}\Big\} .$$

Remark 6: The counterpart of the last inequality in our former analysis was

$$(3.31)\qquad \|\varphi\|^2_{\alpha+1} \le c\|u\|^2_{\alpha+1} + c\|\nabla u\|^2_\alpha + c\|\nabla(w-\chi)\|^2_{-\alpha} .$$

The third step consists in analyzing the terms with $w - \chi$ in (3.30) which still depend on $\varphi$ since $w$ does. Since $\varphi$ and hence $\mu^{-\alpha-1}\varphi$ is in $H_1$ the shift theorem guarantees $w \in H_3$ . We have assumed $m \ge 3$ , i.e. at least quadratic finite elements are used. Therefore we can choose $\chi$ according to Lemma 2 and Corollary 2 with $l = 3$ and get from (3.30)

$$(3.32)\qquad \|\varphi\|^2_{\alpha+1} \le c\, N_\alpha(u)^2 + c\, h^4\|\nabla^3 w\|^2_{-\alpha} .$$

The next section is devoted to the proof of

Lemma 7: Let $\alpha$ be in the range $N/2 < \alpha < (N+1)/2$ . Then for any $w \in \mathring{H}_1 \cap H_3$ with $\Delta w \in \mathring{H}_1$ the a priori estimate

$$(3.33)\qquad \|\nabla^3 w\|_{-\alpha} \le \|\nabla\Delta w\|_{-\alpha} + c\,\rho^{-2}\|\Delta w\|_{-\alpha-1}$$

holds true.

Because of the definition of $w$ (3.23) we find

$$(3.34)\qquad \|\Delta w\|_{-\alpha-1} = \|\varphi\|_{\alpha+1}$$



and

$$(3.35)\qquad \begin{aligned}\|\nabla\Delta w\|_{-\alpha} &= \|\nabla(\mu^{-\alpha-1}\varphi)\|_{-\alpha}\\ &\le c\Big\{\|\varphi\|_{\alpha+3} + \|\nabla\varphi\|_{\alpha+2}\Big\}\\ &\le c\,\rho^{-2}\Big\{\|\varphi\|_{\alpha+1} + \|\nabla\varphi\|_{\alpha}\Big\} .\end{aligned}$$

Now using (3.22) we derive from (3.32)

$$(3.36)\qquad \|\varphi\|_{\alpha+1} \le c_3\frac{h^2}{\rho^2}\|\varphi\|_{\alpha+1} + c\, N_\alpha(u) .$$

In analogy to (3.21) we impose the side constraint

$$(3.37)\qquad \rho \ge \gamma_2 h \quad \text{with} \quad \gamma_2 = \max(\gamma_1, \sqrt{2c_3})$$

on $\rho$ . This leads to

Theorem 8: For $\alpha \in (N/2,(N+1)/2)$ and under the condition $\rho \ge \gamma_2 h$ the $(\alpha+1)$-norm of the Ritz-approximation $\varphi = R_h u$ is bounded by the composed $\alpha$-norm $N_\alpha(.)$ of $u$ itself

$$(3.38)\qquad \|\varphi\|_{\alpha+1} \le c\, N_\alpha(u)$$

with $c$ independent of $h$ , $\rho$ and the point $x_o$ .

## 4. Proof of Lemma 7

The general shift theorem in the theory of elliptic equations includes the two statements

<u>Let</u> $v \in \overset{o}{H}_1 \cap H_2$ . <u>Then</u>

$$\|\nabla^2 v\| \le c\|\Delta v\| \quad . \tag{4.1}$$

<u>Let</u> $v \in \overset{o}{H}_1 \cap H_3$ . <u>Then</u>

$$\|\nabla^3 v\| \le c\left\{\|\nabla\Delta v\| + \|\Delta v\|\right\} \quad . \tag{4.2}$$

A direct consequence is

<u>Lemma 9</u>: <u>Let</u> $v \in \overset{o}{H}_1 \cap H_2$ <u>resp.</u> $v \in \overset{o}{H}_1 \cap H_3$ . <u>Then in weighted norms for</u> $\beta$ <u>arbitrary</u>

$$\|\nabla^2 v\|_\beta \le c\left\{\|\Delta v\|_\beta + \|\nabla v\|_{\beta+1} + \|v\|_{\beta+2}\right\} \quad , \tag{4.3}$$

$$\|\nabla^3 v\|_\beta \le c\left\{\|\nabla\Delta v\|_\beta + \|\Delta v\|_{\beta+1} + \|\nabla v\|_{\beta+2} + \|v\|_{\beta+3}\right\} \tag{4.4}$$

<u>are valid</u>.

<u>Proof</u>: We will give the details only for (4.3), the second case is handled in the same way. For convenience we use $\varepsilon = \beta/2$ . We can rewrite the integrand in

$$\|\nabla^2 v\|_\beta^2 = \sum_{i,k} \iint_\Omega (\mu^{-\varepsilon} v_{ik})^2 \, dx \tag{4.5}$$

by

$$\mu^{-\varepsilon} v_{ik} = (\mu^{-\varepsilon} v)_{ik} - v_i(\mu^{-\varepsilon})_k - v_k(\mu^{-\varepsilon})_i - v(\mu^{-\varepsilon})_{ik} \quad . \tag{4.6}$$

Therefore we get using (2.15)



$$\|\nabla^2 v\|_\beta \le 2\|\nabla^2(\mu^{-\varepsilon} v)\| + c(\|\nabla v\|_{\beta+1} + \|v\|_{\beta+2}) \quad . \tag{4.7}$$

In the similar way it is

$$\Delta(\mu^{-\varepsilon} v) = \mu^{-\varepsilon}\Delta v + 2\nabla v\cdot\nabla(\mu^{-\varepsilon}) + v\Delta\mu^{-\varepsilon} \tag{4.8}$$

leading to

$$\|\Delta(\mu^{-\varepsilon} v)\| \le 2\|\Delta v\|_\beta + c(\|\nabla v\|_{\beta+1} + \|v\|_{\beta+2}) \quad . \tag{4.9}$$

(4.7) together with (4.9) gives (4.3). #

After these preparations we go back to the function $w$ defined by (3.23) and the a priori estimate stated in Lemma 7. By Lemma 9 we have

$$\|\nabla^3 w\|_{-\alpha} \le c\left\{\|\nabla\Delta w\|_{-\alpha} + \|\Delta w\|_{-\alpha+1} + \|\nabla w\|_{-\alpha+2} + \|w\|_{-\alpha+3}\right\} \quad . \tag{4.10}$$

We have at once

$$\|\Delta w\|_{-\alpha+1} \le \rho^{-2}\|\Delta w\|_{-\alpha-1} \quad . \tag{4.11}$$

In order to complete the proof of Lemma 7 we have to show that the sum

$$\|\nabla w\|_{-\alpha+2} + \|w\|_{-\alpha+3} \tag{4.12}$$

is bounded by the right hand side of (3.33). Our choice of $\alpha$ leads to

$$\tfrac{1}{2}(3-N) < -\alpha + 2 < \tfrac{1}{2}(4-N) \quad . \tag{4.13}$$

Therefore the weight $-\alpha+2$ of the term $\nabla w$ in (4.12) is positive in case of $N = 2,3$ dimensions and negative for

$N \geq 4$ dimensions. Moreover, in case of $N = 3$ dimensions we have

$$0 < -\alpha + 2 < \frac{N}{2} - 1 \quad . \tag{4.14}$$

According to this the cases of 2,3 or higher dimension have to be treated separately. This will be clearer because of the following

<u>Lemma 10</u>: <u>Let</u> $v \in \overset{o}{H}_1 \cap H_2$ . <u>Then</u>

( i) <u>for</u> $\beta < 0$ <u>the norms</u> $\|\nabla v\|_\beta$ <u>and</u> $\|v\|_{\beta+1}$ <u>are comparable modulo</u> $\|\Delta v\|_{\beta-1}$ , <u>i.e.</u>

$$\begin{aligned} \|\nabla v\|_\beta &\leq c\left\{\|v\|_{\beta+1} + \|\Delta v\|_{\beta-1}\right\} \quad , \\ \|v\|_{\beta+1} &\leq c\left\{\|\nabla v\|_\beta + \|\Delta v\|_{\beta-1}\right\} \quad , \end{aligned} \tag{4.15}$$

(ii) <u>for</u> $0 < \beta < \frac{N}{2} - 1 (N>2)$ <u>both terms are bounded by the last, i.e.</u>

$$\|\nabla v\|_\beta + \|v\|_{\beta+1} \leq c\|\Delta v\|_{\beta-1} \quad . \tag{4.16}$$

<u>Proof</u>: The identity

$$\|\nabla v\|_\beta^2 = D(v, \mu^{-\beta} v) - \iint_\Omega v \nabla v \cdot \nabla \mu^{-\beta}\, dx \tag{4.17}$$

leads to

$$\|\nabla v\|_\beta^2 = (v. -\Delta v)_\beta + \frac{1}{2} \iint_\Omega v^2 \Delta\mu^{-\beta}\, dx \quad . \tag{4.18}$$

Direct differentiation gives - $r = |x - x_o|$

$$\Delta\mu^{-\beta} = -2\beta\, \mu^{-\beta-2}\left\{N\, \rho^2 + (N-2\beta-2)\, r^2\right\} \quad . \tag{4.19}$$

Thus in case (i) $\Delta\mu^{-\beta}$ is bounded from above and below by



$c\mu^{-\beta-1}$ giving

$$\begin{aligned} \|\nabla v\|_\beta^2 &\leq (v, -\Delta v)_\beta + \bar{c}\|v\|_{\beta+1}^2 \quad , \\ &\geq (v, -\Delta v)_\beta + \underline{c}\|v\|_{\beta+1}^2 \quad . \end{aligned} \tag{4.20}$$

This proves (4.16) since

$$|(v, -\Delta v)_\beta| \leq \delta\|v\|_{\beta+1}^2 + \frac{1}{4\delta}\|\Delta v\|_{\beta-1}^2 \quad . \tag{4.21}$$

#

In case (ii) we have

$$\Delta\mu^{-\beta} \leq -c'\mu^{-\beta-1} \tag{4.22}$$

with a positive constant $c'$ giving

$$\begin{aligned} \|\nabla v\|_\beta^2 + c'\|v\|_{\beta+1}^2 &\leq (v, -\Delta v)_\beta \\ &\leq \frac{1}{2}c'\|v\|_{\beta+1}^2 + \frac{1}{2c'}\|\Delta v\|_{\beta-1}^2 \quad . \end{aligned} \tag{4.23}$$

#

We are now able to give a short proof of Lemma 7 for $N = 3$ dimensions. Because of (4.14) and the second part of Lemma 10 we have

$$\begin{aligned} \|\nabla w\|_{-\alpha+2} + \|w\|_{-\alpha+3} &\leq c\|\Delta w\|_{-\alpha+1} \\ &\leq c\rho^{-2}\|\Delta w\|_{-\alpha-1} \quad . \end{aligned} \tag{4.24}$$

#

Now let us consider the case of $N = 2$ dimensions. We will give an explicite proof of

<u>Corollary 9</u>: <u>Under the assumptions of Lemma 9 the terms</u> $\|v\|_{\beta+2}$ <u>in</u> (4.3) <u>resp.</u> $\|v\|_{\beta+3}$ <u>in</u> (4.4) <u>can be dropped in case of</u> $N = 2$ <u>dimensions, provided</u> $\Delta v \in \overset{o}{H}_1$ .

Remark 7: The restriction to $N = 2$ dimensions is unnecessary. But we will need it only in this case.

Before we give the proof let us finish the proof of Lemma 7. We need now - see (4.12) - a bound of $\|\nabla w\|_{-\alpha+2}$ only. In the present case we have $\frac{1}{2} < -\alpha + 2 < 1$. Let $p_2 > 2$ be fixed with $\alpha - 1 < 2/p_2$. Now we apply Hoelder's inequality with $p = p_2/2 > 1$ and get

$$\|\nabla w\|^2_{-\alpha+2} = \iint \mu^{\alpha-2} |\nabla w|^2 \, dx \tag{4.25}$$
$$\leq \|\nabla w\|^2_{L_{p_2}} \left\{ \iint \mu^{(\alpha-2)q} \, dx \right\}^{1/q}$$

with $1/q = 1 - 1/p_2$. Direct calculation - see the proof of Lemma 5 - leads to

$$\|\nabla w\|_{-\alpha+2} \leq c \, \rho^{-\lambda} \, \|\nabla w\|_{L_{p_2}} \tag{4.26}$$

with

$$\lambda = 1 - \alpha + 2/p_2 \quad . \tag{4.27}$$

Next let $p_1$ be defined by

$$1/p_1 = 1/2 + 1/p_2 \quad . \tag{4.28}$$

By the aid of standard a priori estimates - see MORREY [3] pp.80 and 157 - we get

$$\|\nabla w\|_{L_{p_2}} \leq c \|\nabla^2 w\|_{L_{p_1}} \tag{4.29}$$

and



$$\|\nabla^2 w\|_{L_{p_1}} \leq c \|\Delta w\|_{L_{p_1}} \quad . \tag{4.30}$$

In our case we have $1 < p_1 < 2$. Therefore we may once more apply Hoelder's inequality to

$$\|\Delta w\|^{p_1}_{L_{p_1}} = \iint (\mu^{\alpha+1} \Delta w^2)^{p_1/2} \mu^{-(\alpha+1)p_1/2} \, dx \tag{4.31}$$

this time with $p = 2/p_1$. Similar to above we get

$$\|\Delta w\|_{L_{p_1}} \leq c \, \rho^{-\mu} \|\Delta w\|_{-\alpha-1} \tag{4.32}$$

with

$$\mu = 1 + \alpha - \frac{2}{p_2} \quad . \tag{4.33}$$

The combination of (4.26), (4.29), (4.30), and (4.32) leads to

$$\|\nabla w\|_{-\alpha+2} \leq c \, \rho^{-2} \|\Delta w\|_{-\alpha-1} \tag{4.34}$$

what finishes the proof of Lemma 7 for $N = 2$ dimensions.

We will later on need the trace theorem in weighted norms in the form

<u>Lemma 11</u>: <u>Let</u> $v \in H_1$. <u>Then for</u> $\delta > 0$

$$|v|_{\beta+1/2} \leq \delta \|\nabla v\|_\beta + c(1+\delta^{-1}) \, \|v\|_{\beta+1} \quad . \tag{4.35}$$

<u>Proof</u>: (4.35) is shown by applying the standard trace theorem

$$|V|^2 \leq c \left\{ \|V\|^2 + \|V\| \, \|\nabla V\| \right\} \tag{4.36}$$

to $V = \mu^{-\beta/2-1/4} v$.

<u>Proof of Corollary 9</u>: In $N = 2$ dimensions - we denote the variables by $x,y$ - it is

(4.37)
$$|\nabla^2 v|^2 - |\Delta v|^2 = -2(v_{xx}v_{yy}-v_{xy}^2) = -2\{(v_y v_{xx})_y - (v_y v_{xy})_x\}$$

and therefore

(4.38)
$$\|\nabla^2 v\|_\beta^2 - \|\Delta v\|_\beta^2 = 2 \oint_{\partial\Omega} \mu^{-\beta} v_y \, dv_x + 2 \iint_\Omega v_y\{v_{xx}(\mu^{-\beta})_y - v_{xy}(\mu^{-\beta})_x\} \, dxdy$$

resp.

(4.39) $$\|\nabla^2 v\|_\beta^2 - \|\Delta v\|_p^2 \le 2 \oint_{\partial\Omega} \mu^{-\beta} v_y dv_x + c\|\nabla v\|_{\beta+1} \|\nabla^2 v\|_\beta \; .$$

In order to analyze the boundary integral we introduce the arc length $s$ and the angle $\gamma = \gamma(s)$ between the tangent and the x-axis. Further $v_s, v_n$ denote the tangential and normal differentiation. Because of $v = 0$ on $\partial\Omega$ we have

(4.40) $$v_x = -\sin\gamma \, v_n, v_y = \cos v \, v_n$$

and with $\varkappa = \gamma'$ being the curvature of $\partial\Omega$

(4.41) $$2v_y dv_x = -2\{\varkappa \cos^2\gamma v_n^2 + \sin\gamma\cos\gamma \, v_n v_{ns}\} \, ds \; .$$

We insert this in the boundary integral and apply partial integration becuase of $v_n v_{ns} = (v_n)^2{}_s/2$ . Then we get

(4.42) $$|2 \oint_{\partial\Omega} \mu^{-\beta} v_y dv_x| \le c|\nabla v|_{\beta+1/2}^2 \; .$$

With the help of Lemma 11 then (4.39) leads to



(4.43) $$\|\nabla^2 v\|_\beta^2 - \|\Delta v\|_p^2 \le 2\delta\|\nabla^2 v\|_\beta^2 + \frac{c}{\delta} \|\nabla v\|_{\beta+1}^2 \; .$$

This proves (4.3) without the last term on the right hand side.

In proving the second part of Corollary 9 we will skip some of the details. In the corresponding way to above we get the counterpart of (4.39).

(4.44)
$$\|\nabla^3 v\|_\beta^2 - \|\nabla\Delta v\|_p^2 \le 2 \oint_{\partial\Omega} \mu^{-\beta}(v_{yy}-v_{xx}) \, dv_{xy} + c\|\nabla^2 v\|_{\beta+1} \|\nabla^3 v\|_\beta \; .$$

On $\partial\Omega$ we have for $v$ arbitrary with the abbreviations $s := \sin\gamma$ , $c := \cos v$

(4.45)
$$\begin{aligned} v_{ss} &= c^2 v_{xx} + 2sc \, v_{xy} + s^2 v_{yy} + \varkappa v_n \, , \\ v_{ns} &= -sc \, v_{xx} + (c^2-s^2) v_{xy} + sc \, v_{yy} - \varkappa v_s \, , \\ v_{nn} &= s^2 v_{xx} - 2sc \, v_{xy} + c^2 v_{yy} \; . \end{aligned}$$

The condition $v = 0$ implies $v_s = v_{ss} = 0$ . In addition $\Delta v = 0$ implies $v_{nn} = \varkappa v_n$ . Therefore we derive

(4.46)
$$\begin{aligned} v_{yy} - v_{xx} &= 2\varkappa \cos 2\gamma \, v_n + 2\sin 2\gamma \, v_{ns} \\ 2v_{xy} &= -2\varkappa \sin 2\gamma \, v_n + 2\cos 2\gamma \, v_{ns} \; . \end{aligned}$$

Similar to above we then get

(4.47) $$|\oint_{\partial\Omega} \mu^{-\beta}(v_{yy}-v_{xx}) \, dv_{xy}| \le c\{|\nabla^2 v|_{\beta+1/2}^2 + |\nabla v|_{\beta+1/2}^2\}$$

and therefore with Lemma 11

$$(4.48)\qquad \|\nabla^3 v\|_\beta^2 - \|\nabla\Delta v\|_\beta^2 \le 2\delta\|\nabla^3 v\|_\beta^2 + \frac{c}{\delta}\left\{\|\nabla^2 v\|_{\beta+1}^2 + \|\nabla v\|_{\beta+2}^2\right\}$$

resp.

$$(4.49)\qquad \|\nabla^3 v\|_\beta \le c\left\{\|\nabla\Delta v\|_\beta + \|\nabla^2 v\|_{\beta+1} + \|\nabla v\|_{\beta+2}\right\} .$$

Now we have to apply the first part of Corollary 9 to the second term on the right hand side of (4.49) #

Remark 8: Above we derived the a priori estimates needed for functions sufficiently smooth only. For instance (4.37) holds only for functions having third derivatives. By compactness arguments the validity of the estimates for functions with the stated regularity is shown.

The case of $N \ge 4$ dimensions hardly is of practical importance. Therefore we give only an outline of the proof for this case. In view of Lemma 10 and because of (4.13) it is only necessary to bound $\|w\|_{-\alpha+3}$ in terms of $\|\Delta w\|_{-\alpha-1}$, i.e. to find an upper bound of

$$(4.50)\qquad \lambda(\Omega) = \sup \|w\|_{-\alpha+3}^2 / \|\Delta w\|_{-\alpha-1}^2$$

where the supremum is to be taken over all $v \in \overset{o}{H}_1 \cap H_2$. Obviously the supremum is attained for an eigenfunction of the problem

$$(4.51)\qquad \begin{aligned} \Delta(\mu^{\alpha+1}\Delta v) &= \lambda^{-1}\mu^{\alpha-3} v \quad \text{in} \quad \Omega , \\ v = \Delta v &= 0 \quad \text{on} \quad \partial\Omega . \end{aligned}$$



In this way we ask for a lower bound of the smallest eigenvalue of problem (4.51). By standard arguments the monotonicity of $\lambda$ with respect to the domain, i.e. $\lambda(\Omega_1) \le \lambda(\Omega_2)$ in case of $\Omega_1 \subseteq \Omega_2$, is shown. Therefore an upper bound for $\lambda(\Omega)$ is given by the corresponding $\lambda$ for the ball with center in $x_o$ and radius $d$ = diameter $(\Omega)$. The eigenfunction corresponding to the lowest eigenvalue then depends only on $r = |x-x_o|$ (or at least one does). Using the representation

$$(4.52)\qquad \nabla v \sim v' = r^{1-N}\int_0^r s^{N-1}\,\Delta v\,ds$$

we get without difficulties

$$(4.53)\qquad \|\nabla v\|_{-\alpha+2} \le c\,\rho^{-2}\|\Delta v\|_{-\alpha-1}$$

which in view of Lemma 10 bounds $\|v\|_{-\alpha+3}$ in the same way.

## 5. The Boundedness of the Ritz Approximation in Hoelder Spaces

The Laplacian like any elliptic operator is not one to one with respect to the spaces $C^k = C^k(\Omega)$ consisting of functions having continuous derivatives up to order $k$ in $\bar{\Omega}$. We will also abbreviate $C = C^0$ and denote by $\overset{o}{C}$ the space of continuous functions vanishing on the boundary $\partial\Omega$. Of course the image $f = \Delta u$ of any $u \in \overset{o}{C} \cap C^{k+2}$ $(k \geq 0)$ is in $C^k$ but to $f \in C^k$ there may not be an original $u \in \overset{o}{C} \cap C^{k+2}$ as is demonstrated in two dimensions by the counterexample

$$(5.1) \qquad u = (x^2-y^2)|\ln(x^2+y^2)|^{1/2}$$

with $\Omega$ the unit sphere.

The situation is changed in case of Hoelder- (resp. Lipschitz-) spaces. These spaces, denoted by $C^{k.\lambda} = C^{k.\lambda}(\Omega)$ with $\lambda$ according to $0 < \lambda \leq 1$, consist of all functions k-times continuously differentiable such that the highest derivatives are Hoelder-continuous to the exponent $\lambda$. In $C^{k.\lambda}$ a norm is given by

$$(5.2) \qquad \|v\|_{C^{k.\lambda}} = \sum_{|\xi|\leq k} \|D^{\xi}v\|_{L_\infty} + \sum_{|\xi|=k} \left[D^{\xi}v\right]_\lambda$$

with

$$(5.3) \qquad [w]_\lambda = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda} \quad .$$

Equipped with this norm $C^{k.\lambda}$ is a Banach space. The Laplacian is a one to one mapping of $\overset{o}{C} \cap C^{k+2.\lambda}$ into



$C^{k.\lambda}$. Especially

$$(5.4) \qquad \|u\|_{C^{k+2.\lambda}} = \|(-\Delta)^{-1}f\|_{C^{k+2.\lambda}} \leq c\|f\|_{C^{k.\lambda}} \quad .$$

Such an a priori estimate is referred to 'Schauder estimate'.

The aim of this section is the proof of corresponding estimates with $u$ replaced by $\varphi = R_h u$ being the Ritz approximation.

A first result in this direction is more or less a direct consequence of Theorem 8. See the proof of Lemma 5 - the right hand side of (3.38) is bounded by

$$(5.5) \qquad N_\alpha(u) \leq c\, \rho^{-\alpha+\frac{N}{2}} h^{-1}\|u\|_{L_\infty} \quad .$$

By Lemma 6 we know

$$(5.6) \qquad \|\varphi\|_{L_\infty} \leq c\, \rho^{\alpha+1} h^{-\frac{N}{2}} \sup_{x_0} \|\varphi\|_{\alpha+1} \quad .$$

Besides (3.37) $\rho$ is arbitrary. Now we fix $\rho = \gamma_2 h$ and get

$$(5.7) \qquad \|\varphi\|_{L_\infty} = \|R_h u\|_{L_\infty} \leq c\|u\|_{L_\infty} \quad .$$

This gives

**Theorem 12:** The Ritz operator is bounded as mapping of $\overset{o}{C}$ into itself.

The spaces $C^{k.\lambda}$ are compactly embedded on $C$. There is a general principle to bound the norm in $C^{k.\lambda}$ of a linear projection operator by means of the norm in $C$ which we will discuss now. The situation is that we have two Banach spaces $X_1, X_2$ (with norms $\|\cdot\|_1$, $\|\cdot\|_2$) with $X_2$ compactly embedded in $X_1$. Further we have a collection $\{S_h \mid 0 < h \leq 1\}$ of subspaces of $X_2$. Let approximation- and inverse-quantities $\sigma_h$ and $\tau_h$ be introduced according to

(A) To any $y \in X_2$ there is an $\eta \in S_h$ such that simultaneously

$$(5.8)\qquad \begin{aligned} \|y-\eta\|_1 &\leq \sigma_h \|y\|_2 \quad , \\ \|\eta\|_2 &\leq c_1 \|y\|_2 \end{aligned}$$

is valid with $c_1$ independent of $h$.

(I) For any $x \in S_h$ a Bernstein type inequality holds

$$(5.9)\qquad \|x\|_2 \leq \tau_h \|x\|_1 \quad .$$

We will 'say' the collection $\{S_h\}$ fulfills the AI-condition if

$$(5.10)\qquad K := \sup_h \sigma_h \tau_h < \infty \quad .$$

<u>Remark 9</u>: Under 'reasonable' assumptions $\sigma_h$ will tend to zero with $h$. For finite dimensional spaces $S_h$ the quantities $\tau_h$ are finite since then any two norms are equivalent. With $h \to 0$ resp. $\dim(S_h) \to \infty$ then $\tau_h$ will also tend to infinity. The AI-condition just balances this.

The mentioned principle is



<u>Lemma 13: Let $X_1, X_2$ be as described above and $\{S_h\}$ a collection of subspaces of $X_2$. Further let $\{P_h : X_1 \to S_h\}$ be a collection of linear projection operators of $X_1$ onto $S_h$ which are uniformly bounded as mappings of $X_1$ into itself, i.e.</u>

$$(5.11)\qquad \|P_h\|_1 = \sup_{y \neq 0} \frac{\|P_h y\|_1}{\|y\|_1} \leq p_1$$

<u>with $p_1$ independent of $h$. If $\{S_h\}$ fulfills the AI-condition then $\{P_h\}$ as mapping of $X_2$ into itself is uniformly bounded with</u>

$$(5.12)\qquad \|P_h\|_2 = \sup_{y \neq 0} \frac{\|P_h y\|_2}{\|y\|_2} \leq p_2 := (c_1 + 3K)\, p_1 \quad .$$

<u>Proof</u>: Because of $X_2 \subseteq X_1$ and $S_h \subseteq X_2$ of course $P_h$ is a linear projection of $X_2$ into itself. Let $y \in X_2$ be given and $\eta \in S_h$ be chosen according to (5.8). Then

$$(5.13)\qquad \|P_h y\|_2 \leq \|P_h y - \eta\|_2 + c_1 \|y\|_2 \quad .$$

Since $P_h y - \eta$ is an element of $S_h$ we may apply (5.9) getting

$$(5.14)\qquad \begin{aligned} \|P_h y\|_2 &\leq \tau_h \|P_h y - \eta\|_1 + c_1 \|y\|_2 \\ &\leq \tau_h \left\{ \|P_h y - y\|_1 + \|y-\eta\|_1 \right\} + c_1 \|y\|_2 \quad . \end{aligned}$$

Now we use the inequality

$$(5.15)\qquad \|y - P_h y\|_1 \leq (1 + \|P_h\|_1) \inf_{\tilde\eta \in S_h} \|y - \tilde\eta\|_1$$

the proof of which - in order to give a selfcontained presentation - is as follows: Let $\tilde{\eta} \in S_h$ be arbitrary. Because of $P_h\tilde{\eta} = \tilde{\eta}$ we have

$$\|y-P_hy\|_1 = \|y-\tilde{\eta} - P_h(y-\tilde{\eta})\|_1 \le (1+\|P_h\|_1)\ \|y-\tilde{\eta}\|_1 \quad . \tag{5.16}$$

In (5.15) resp. (5.16) we may use on the right hand side $\tilde{\eta} = \eta$.

Because of the assumption (5.11) we get from (5.16)

$$\|y-P_hy\|_1 \le (1+p_1)\ \|y-\eta\|_1 \tag{5.17}$$

and in this way from (5.14)

$$\|P_hy\|_2 \le (2+p_1)\ \tau_h\|y-\eta\|_1 + c_1\|y\|_2 \quad . \tag{5.18}$$

Finally using (5.8) we come to

$$\|P_hy\|_2 \le \{(2+p_1)\ \sigma_h\ \tau_h + c_1\}\ \|y\|_2 \quad . \tag{5.19}$$

The norm of any projection operator is bounded from below by 1 . Therefore we can also bound

$$p_2 \le (3K+c_1)\ p_1 \tag{5.20}$$

which is more convenient.

Remark 10: Lemma 13 first was stated in NITSCHE [5] .

It remains to prove

Lemma 14: Assume $S_h \subseteq C^k$ . Then with $X_1 = \overset{\circ}{C}$ and $X_2 = C^{k.\lambda}$ the finite element spaces $\overset{\circ}{S}_h$ fulfill the AI-condition.



The consequence is the final result:

Theorem 15: Assume $S_h \subseteq C^k$ . Then the Ritz operator is bounded as mapping of $C^{k.\lambda}$ into itself.

Proof of Lemma 14: The finite elements discussed in Section 2 are only in $C$ . We will give the proof only for the case $k = 0$ . The case $k \ge 1$ follows the same lines and is omitted here in order to avoid the introduction of finite elements with higher smoothness. We will show that the standard interpolation will have the properties needed. Especially we will show

$$\sigma_h \le c\ h^{\lambda}\ ,\ \tau_h \le c\ h^{-\lambda} \quad . \tag{5.21}$$

First we prove the estimate for $\tau_h$ . Similar to Lemma 3 we have for $\chi \in S_h$

$$\|\nabla\chi\|'_{L_\infty} = \max_{\Delta\in\Gamma_h} \|\nabla\chi\|_{L_\infty(\Delta)} \le c\ h^{-1}\|\chi\|_{L_\infty} \quad . \tag{5.22}$$

Now let $x,y \in \Omega$ be given. In case of $|x-y| \ge h$ we have trivially

$$\frac{|\chi(x)-\chi(y)|}{|x-y|^{\lambda}} \le 2h^{-\lambda}\|\chi\|_{L_\infty} \quad . \tag{5.23}$$

In case of $|x-y| < h$ we come from

$$|\chi(x)-\chi(y)| \le |x-y|\ \|\nabla\chi\|'_{L_\infty} \tag{5.24}$$

to

$$\frac{|\chi(x)-\chi(y)|}{|x-y|^{\lambda}} \le c\, h^{-\lambda} \left\{\frac{|x-y|}{h}\right\}^{1-\lambda} \|\chi\|_{L_\infty}$$

(5.25)

$$\le c\, h^{-\lambda} \|\chi\|_{L_\infty} \quad .$$

Now we turn over to the estimation of $\sigma_h$ . Referring to **CIARLET [1], pp.43 for details there exists to any $\Delta \in \Gamma_h$** a set of points $\{P_j = P_j^{\Delta} \mid j = 1,\dots,J\}$ ($J = \dim \mathbb{P}_{m-1}$ , the space of polynomials of degree less than $m$ ) with the following properties:

( i) The conditions

(5.26) $$p^{\Delta}(P_j) = r_j \quad \text{for} \quad j = 1,\dots J$$

define uniquely a polynomial $p^{\Delta}$ of degree less than $m$ .

(ii) If $r_j = r_j^{\Delta}$ coincide with the values in $P_j^{\Delta}$ of a function $v$ continuous in $\Omega$ then the function $\chi$ defined by

(5.27) $$\chi_{|\Delta} = p^{\Delta}$$

is continuous in $\Omega$ .

Now let $p$ be the restriction to a $\Delta \in \Gamma_h$ fixed of the interpolation of a function $v \in C^{0,\lambda}$ . For convenience let - possibly after a translation - the origin coincide with one of the corners of $\Delta$ , say $P_1$ . Then $p$ has the structure

(5.28) $$p(x) = \sum_{|\xi|<m} x^{\xi} \; c_{\xi}(v)\, h^{-|\xi|}$$

with



(5.29) $$x^{\xi} = x_1^{\xi_1} \cdot \ldots \cdot x_N^{\xi_N}$$

and

(5.30) $$c_{\xi}(v) = \sum_{j=1}^{J} c_{\xi}^{j}\, v(P_j) \quad .$$

The $\varkappa$-regularity of the subdivision $\Gamma_h$ leads to the uniform boundedness of the $c_{\xi}^{j}$ independent of $h$ .

Since the function $v = 1$ is reproduced by the interpolation we have

(5.31) $$\sum_{j=1}^{J} c_{\xi}^{j} = \begin{cases} 1 & \text{for } |\xi| = 0 , \\ 0 & \text{for } |\xi| \ge 1 . \end{cases}$$

This gives on the one hand

(5.32) $$c_o(v) = v(P_1)$$

and on the other hand for $c_{\xi}$ with $|\xi| \ge 1$ a representation

(5.33) $$c_{\xi}(v) = \sum_{j_1, j_2} \tilde{c}_{\xi}^{\,j_1, j_2}(v(P_{j_1}) - v(P_{j_2}))$$

with some $\tilde{c}_{\xi}^{\,j_1,j_2}$ also uniformly bounded. With $x \in \Delta$ we get

(5.34) $$v(x) - p(x) = v(x) - v(P_1) - \sum_{1\le|\xi|<m} \frac{x^{\xi}}{h^{|\xi|}}\, c_{\xi}(v).$$

For $v \in C^{0,\lambda}$ we have

(5.35) $$|v(x)-v(P_1)| \le [v]_{\lambda}\, h^{\lambda} \quad .$$

Because of $|x| \le h$ in $\Delta$ we get with (5.33)

$$\Big| \sum_{1\le|\xi|<m} \{\ldots\} \Big| \le c \max |v(P_{j_1})-v(P_{j_2})| \tag{5.36}$$
$$\le c[v]_\lambda \, h^\lambda \quad .$$

This proves the first part of the approximation property (5.8) with $\sigma_h \le c\, h^\lambda$ .

In order to prove the second part we consider firstly two points $x,y$ contained in one of the simplices $\Delta$ . Then with $d = |x-y|$ we have $d \le h$ and

$$p(x) - p(y) = \sum_{1\le|\xi|<m} h^{-|\xi|}(x^\xi - y^\xi)\, c_\xi(v) \quad . \tag{5.37}$$

Because of

$$|x^\xi - y^\xi| \le c\, d\, h^{|\xi|-1} \tag{5.38}$$

and - see (5.33)

$$|c_\xi(v)| \le c\, h^\lambda [v]_\lambda \tag{5.39}$$

we get

$$|p(x)-p(y)| \le c\, d\, h^{\lambda-1}[v]_\lambda \le c|x-y|^\lambda [v]_\lambda \quad . \tag{5.40}$$

In case of $d = |x-y| < h$ but $x \in \Delta_1$ and $y \in \Delta_2$ with $\Delta_1 \ne \Delta_2$ the segment connecting $x$ and $y$ intersects only a finite number of $\Delta \in \Gamma_h$ because of the $\varkappa$-regularity. By estimates similar to above we get for the interpolation $\chi = I_h v$ also then

$$|\chi(x)-\chi(y)| \le c|x-y|^\lambda [v]_\lambda \quad . \tag{5.41}$$

In case of $d = |x-y| > h$ and $x \in \Delta_1$ , $y \in \Delta_2$ we se-



lect two corners $P_x, P_y$ of $\Delta_1, \Delta_2$ . Then we have

$$\chi(x) - \chi(y) = (\chi(x)-\chi(P_x)) + (\chi(P_x)-\chi(P_y)) + \tag{5.42}$$
$$+ (\chi(P_y)-\chi(y)) \quad .$$

According to the choice of $P_x, P_y$ we have $|x-P_x| \le h$ and $|y-P_y| \le h$ and therefore

$$|\chi(x)-\chi(P_x)| \le c\, h^\lambda [v]_\lambda \quad , \tag{5.43}$$
$$|\chi(y)-\chi(P_y)| \le c\, h^\lambda [v]_\lambda \quad .$$

Since $\chi$ is the interpolation on $v$ we have

$$|\chi(P_x)-\chi(P_y)| = |v(P_x)-v(P_y)| \tag{5.44}$$
$$\le |P_x-P_y|^\lambda [v]_\lambda \quad .$$

We have $d \ge h$ and $|P_x-P_y| \le d + 2h \le 3d$ . In this way also the second part of (5.8) is proven.

## 6. General Second Order Elliptic Equations

In Sections 3 and 4 we presented the $L_\infty$-analysis of the Ritz procedure in case of the Laplacian being the prototype of an elliptic differential operator. The same results hold in the general case with $-\Delta$ replaced by

$$Au = -a^{ik}u_{ik} + b^i u_i + du \quad . \tag{6.1}$$

Remark 11: Throughout this section we adopt the summation convention. Lower indices indicate differentiation with respect to the corresponding variable.

The assumptions regarding the coefficients are:

(a.1) Ellipticity: There is a constant $\underline{q} > 0$ such that for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$a^{ik}\xi_i\xi_k \geq \underline{q} \sum_1^N \xi_i^2 \tag{6.2}$$

holds true.

(a.2) Regularity: The coefficients $a^{ik}$, $b^i$, and $d$ fulfill

$$a^{ik} \in C^{2.1} \ , \quad b^i \in C^{1.1} \ , \quad d \in C^{0.1} \quad . \tag{6.3}$$

The letter $\overline{q}$ is used as an upper bound of all the corresponding norms.

Remark 12: Assumption (a.2) guarantees that the coefficients of the formal adjoint operator $A^*$ defined by

$$A^*v = -(a^{ik}v)_{ik} - (b^i v)_i + dv \tag{6.4}$$

fulfills also (a.2).



The weak formulation of the boundary value problem

$$\begin{aligned} Au &= f \quad \text{in} \quad \Omega \ , \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{6.5}$$

is

Find $u \in \overset{o}{H}_1$ such that

$$a(u,v) = (f,v) \tag{6.6}$$

holds for all $v \in \overset{o}{H}_1$

with $a(\ldots)$ defined by

$$a(v,w) = \iint_\Omega \left\{a^{ik}v_i w_k + b^i v_i w + d\, vw\right\} dx \quad . \tag{6.7}$$

Coresspondingly the (generalized) Ritz approximation $\varphi = R_h u \in \overset{o}{S}_h$ is characterized by the relation

$$a(\varphi,\chi) = (f,\chi) \quad \text{for} \quad \chi \in \overset{o}{S}_h \quad . \tag{6.8}$$

In this generality the function $u$ defined by (6.6) resp. $\varphi$ defined by (6.8) may not exist or may not be unique. Therefore necessarily we assume

(a.3) Existence: The problem (6.5) resp. (6.6) possesses a unique solution for $f$ being arbitrary.

By an argument due to SCHATZ [7] there is an $h_o > 0$ such that for $h \leq h_o$ the Ritz approximation $\varphi$ (6.8) is also unique.

Now we repeat the arguments of Sections 3 and 4. The counterpart of (3.8) in the form

$$(6.9) \qquad \|\nabla\varphi\|_\alpha^2 \le c\{a(\varphi,\mu^{-\alpha}\varphi) + \|\varphi\|_{\alpha+1}^2\}$$

is a direct consequence of Gårding's inequality

$$(6.10) \qquad a(v.v) \ge \hat{q}\|\nabla v\|^2 - \Lambda\|v\|^2$$

for any $v \in \overset{o}{H}_1$ with $\hat{q} > 0, \Lambda$ depending only on $\underline{q}, \overline{q}$ .

Remark 13: The constants c - see the beginning of Section 2 - may depend in addition on (v) the bounds $\underline{q}, \overline{q}$ of the assumptions (a.1). (a.2).

Following the lines of Section 3 we get from (6.9) also now the final estimate (3.22) of Step 1.

The auxiliary function, w - see (3.23) - is defined this time by

$$(6.11) \qquad \begin{aligned} A^* w &= \mu^{-\alpha-1}\varphi \quad \text{in} \quad \Omega \ . \\ w &= 0 \qquad\quad \text{on} \quad \partial\Omega \ . \end{aligned}$$

The estimates leading to (3.32) are derived in the same way as before.

Since the shift theorems (4.1), (4.2) are valid with $-\Delta$ - the Laplacian - replaced by the operator $A$ Lemma 9 is valid with $-\Delta$ replaced by $A$ on the right hand sides. As before it remains to find bounds of the terms in (4.12).

Following the lines of Section 4 we consider the case of $N = 3$ dimensions firstly. In the general case the second



assertion of Lemma 10 has to be changed by the estimate

$$(6.12) \qquad \|\nabla v\|_\beta + \|v\|_{\beta+1} \le c\{\|Av\|_{\beta-1} + \|v\|_\beta\} \quad .$$

The last term on the right hand side may be treated as was done in the sequence (4.25) to (4.34), the details are omitted. In this way the case of $N = 3$ dimensions is settled.

In accordance to (6.12) the a priori estimates stated in Corollary 9 have to be modified:

<u>Corollary 9A</u>: <u>Let</u> $v \in \overset{o}{H}_1 \cap H_2$ <u>resp.</u> $v \in \overset{o}{H}_1 \cap H_3$ <u>and in addition</u> $Av \in \overset{o}{H}_1$ . <u>Then in weighted norms for</u> $\beta$ <u>arbitrary and</u> $N = 2$ <u>dimensions</u>

$$(6.13) \qquad \|\nabla^2 v\|_\beta \le c\{\|Av\|_\beta + \|\nabla v\|_{\beta+1} + \|v\|_{\beta+1}\} \quad ,$$

$$(6.14) \qquad \|\nabla^3 v\|_\beta \le c\{\|\nabla Av\|_\beta + \|Av\|_{\beta+1} + \|\nabla v\|_{\beta+2} + \|v\|_{\beta+2}\} \ .$$

Having these shift theorems the final proof of Lemma 7 in case of a general second order elliptic differential equation follows the lines of Section 4.

We will not give all the details in order to prove Lemma 9A but concentrate ourselves on the essential point. What is needed are the counterparts of (4.37) resp. (4.39) and of (4.44). By (4.37) the square sum of the second derivatives is bounded by the square of the Laplacian modulo lower order terms and a divergence term of products of first and second derivatives. In order to get the counterparts we make use of

<u>Lemma 16</u>: <u>Let</u> $(a^{ik})$ <u>be a positive definite and symmetric matrix according to</u> (6.2) <u>and let</u> $(b_{ik})$ <u>be a second order tensor. Then</u>

(6.15) $$\underline{q}^2 \sum_{i,k=1}^{N} b_{ik}^2 \leq a^{ik}a^{rs}b_{ir}b_{ks} \quad .$$

<u>Proof:</u> Let $\{z_i^\alpha | \alpha = 1,\ldots,N\}$ be an orthornomal set of eigen-vectors of the matrix $(a^{ik})$ and $\{\lambda^\alpha\}$ be the corresponding set of eigen-values, i.e.

(6.16) $$a^{ik}z_i^\alpha = \lambda^\alpha z_k^\alpha \quad \text{for } \alpha = 1,\ldots,N \quad .$$

The orthogonality conditions

(6.17) $$z_i^\alpha z_i^\beta = \delta^{\alpha\beta}$$

give rise to

(6.18) $$\sum_\alpha z_i^\alpha z_k^\alpha = \delta_{ik}$$

with $\delta^{\alpha\beta}, \delta_{ik}$ denoting the Kronecker symbol.

<u>Remark 14</u>: In the following the summation convention is not to be applied with respect to Greek letters.

The matrix $(a^{ik})$ admits the representation

(6.19) $$a^{ik} = \sum_\alpha \lambda^\alpha z_i^\alpha z_k^\alpha \quad .$$

Then we get

(6.20) $$a^{ik}a^{rs}b_{ir}b_{ks} = \sum_{\alpha,\beta} \lambda^\alpha\lambda^\beta z_i^\alpha z_k^\alpha z_r^\beta z_s^\beta b_{ir} b_{ks} = \sum_{\alpha,\beta} \lambda^\alpha\lambda^\beta |\tilde{b}^{\alpha\beta}|^2$$



with

(6.21) $$\tilde{b}^{\alpha\beta} = z_i^\alpha z_k^\beta b_{ik} \quad .$$

Because of $\lambda^\alpha \geq \underline{q}$ we get therefore

(6.22) $$a^{ik}a^{rs}b_{ir}b_{ks} \geq \underline{q}^2 \sum_{\alpha,\beta} |\tilde{b}^{\alpha\beta}|^2 = \underline{q}^2 \sum_{\alpha,\beta} z_i^\alpha z_k^\beta b_{ik} z_r^\alpha z_s^\beta b_{rs} \quad .$$

With the help of (6.18) we come from the last inequality to (6.15). #

Now we apply (6.15) with $b_{ik} = v_{ik}$. Then we get

(6.23) $$\underline{q}^2\|\nabla^2 v\|_\beta^2 \leq \iint_\Omega \mu^{-\alpha}(a^{ik}a^{rs}v_{ir}v_{ks})\, dx \quad .$$

Besides of lower order terms the right hand side differs from $\|Av\|_\beta^2$ by the weighted integral of the difference

(6.24) $$\begin{aligned} a^{ik}a^{rs}v_{ir}v_{ks} - (a^{ik}v_{ik})(a^{rs}v_{rs}) &= \\ = (a^{ik}a^{rs}v_i v_{ks})_r - (a^{ik}a^{rs}v_i v_{rs})_k &- \\ - (a^{ik}a^{rs})_r\, v_i v_{ks} + (a^{ik}a^{rs})_k\, v_i v_{rs} & \quad . \end{aligned}$$

This leads to an inequality of the structure

(6.25) $$\begin{aligned} \underline{q}^2\|\nabla^2 v\|_\beta^2 \leq \|Av\|_\beta^2 &+ \\ + c\left\{\|\nabla^2 v\|_\beta \|\nabla v\|_{\beta+1} + \|\nabla v\|_{\beta+1}^2 + \|v\|_{\beta+1}^2\right\} &+ \\ + \int_{\partial\Omega} \mu^{-\beta} a^{ik}a^{rs}v_i\left\{v_{ks}n_r - v_{rs}n_k\right\} ds & \quad . \end{aligned}$$