Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations

Klaus BRAUN

Institut für Angewandte Mathematik,
Albert-Ludwigs-Universität,
Hermann-Herder-Str. 10,
7800 Freiburg i.Br. FRG

Reprinted from the JAPAN JOURNAL OF APPLIED MATHEMATICS
Vol. 3, No. 1, pp. 59–72, June 1986

Japan J. Appl. Math., 3 (1986), 59–72

Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations

Klaus BRAUN

Institut für Angewandte Mathematik,
Albert-Ludwigs-Universität,
Hermann-Herder-Str. 10,
7800 Freiburg i.Br. FRG

Received April 25, 1985
Revised October 31, 1985

The Ritz method for strong elliptic pseudo-differential equations is discussed. 'Optimal' local error estimates are derived if the underlying 'approximation-spaces' are finite elements. The analysis covers simultaneously pseudo-differential operators of positive and negative order. In case of positive order an additional regularity assumption for the 'approximation-spaces' is needed.

Key words: pseudo-differential equations, Ritz method, interior estimates, super-approximability

0. Introduction

Let the linear equation

(0.1) $$Au=f$$

be given in a Hilbert-space H with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. The operator A is assumed to have the properties

(0.2)
i) A positive, i.e. $(Au, u)>0$ for $u\neq 0$,
ii) A symmetric, i.e $(Au, v)=(u, Av)$

for $u, v\in D(A)$. Then

(0.3) $$a(u, v):=(Au, v)$$

defines an inner product in $D(A)$. The corresponding norm will be denoted by

(0.4) $$|||u|||:=a(u, u)^{1/2}\,.$$

In order to apply the Ritz method we need a closed 'approximation-space'

(0.5) $$S=S_h\subseteq H_A:=\overline{D(A)}^{|||\cdot|||}\,.$$

The domain of definition of $a(\cdot, \cdot)$ can be extended to $H_A\times H_A$. The Ritz approximation $u_h := R_h u \in S_h$ is defined by

$$(0.6) \qquad a(u_h, \chi) = (f, \chi) \quad \text{for all} \quad \chi \in S_h .$$

Because of

$$(0.7) \qquad a(u, v) = (f, v) \quad \text{for all} \quad v \in D(A)$$

primarily and hence also for $v \in H_A$ we get the defining relation

$$(0.8) \qquad a(u - u_h, \chi) = 0 \quad \text{for all} \quad \chi \in S_h .$$

This shows:

> The error $u - u_h$ is orthogonal to S_h with respect to the inner product $a(\cdot, \cdot)$. Therefore the Ritz method is best approximating in the norm of H_A, i.e.

$$(0.9) \qquad |||u - u_h||| = \inf_{\chi \in S_h} |||u - \chi||| .$$

In order to analyze the error $u - u_h$ further properties of the operator A resp. the space H_A are needed.

We will give two illustrations. Regarding the notations we refer to Gilbarg-Trudinger [6].

Example 1. $H = L_2(\Omega)$ with $\Omega \subseteq \boldsymbol{R}^N$ a bounded domain and the boundary $\partial\Omega$ sufficiently smooth.

$$(0.10) \qquad Au := -\Delta u \quad \text{and} \quad D(A) = \dot{W}_2^2(\Omega) := \mathring{W}_2^1(\Omega) \cap W_2^2(\Omega) .$$

We introduce a Hilbert-scale in the following way: Let $\{v_i, \lambda_i\}$ be the orthonormal set of eigen-pairs of A, i.e.

$$(0.11) \qquad \begin{aligned} -\Delta v_i &= \lambda_i v_i && \text{in} \quad \Omega \\ v_i &= 0 && \text{on} \quad \partial\Omega . \end{aligned}$$

The Hilbert-spaces $\{H_\beta \mid \beta \in \boldsymbol{R}\}$ are spanned by the functions with a finite β-norm defined by

$$(0.12) \qquad \|z\|_\beta^2 := \sum \lambda_i^\beta z_i^2 \quad \text{with} \quad z_i := (z, v_i) .$$

We have the inclusions

$$(0.13) \qquad D(A) \subseteq H_A = H_1 = \mathring{W}_2^1(\Omega) \subseteq L_2(\Omega) .$$

Example 2. $H = L_2^{\#}(\Gamma)$ with $\Gamma = S^1(\boldsymbol{R}^2)$, i.e. Γ is the boundary of the unit sphere. Then H is the space of L_2-integrable periodic functions in $\boldsymbol{R}$.

$$(0.14) \qquad (Au)(x) := \oint k(x - y) u(y) dy \quad \text{and} \quad D(A) = H$$

with

(0.15) $$k(y) := -\ln\left|2\sin\frac{y}{2}\right|.$$

With the help of the Fourier coefficients v_ν of a 2π-periodic function v defined by

(0.16) $$v_\nu := \frac{1}{2\pi}\oint v(x)e^{-i\nu x}dx$$

we may introduce for real β the norms

(0.17) $$\|v\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |v_\nu|^2 .$$

The Hilbert-spaces $H_\beta = H_\beta(\Gamma)$ are defined similar to the above. The Fourier coefficients of the convolution Au (0.14) are

(0.18) $$(Au)_\nu = k_\nu \cdot u_\nu = \frac{1}{2|\nu|} \cdot u_\nu .$$

This time we have (see Hsiao-Wendland [8])

(0.19) $$D(A) \subseteq H_A = H_{-1/2}(\Gamma) .$$

The meaning of local convergence will be demonstrated in case of Example 1. We use isoparametric finite elements

(0.20) $$S_h \subseteq \mathring{W}_2^1(\Omega)$$

which are piecewise polynomials of degree less than t (see Zlàmal [20]). The index $h \in (0, 1]$ is a measure of the underlying subdivision. In Nitsche-Schatz [14] it is shown:

> Let $u \in \dot{W}_2^2(\Omega)$ be the solution of (0.10) with the additional regularity $u \in W_2^\tau(\Omega_2)$ for $\tau \leq t$ and some subdomain $\Omega_2 \subseteq \Omega$. In a proper subdomain $\Omega_1 \Subset \Omega_2$ the error estimate

(0.21) $$\|u - u_h\|_{W_2^\kappa(\Omega_1)} \leq ch^{\tau-\kappa}\{\|u\|_{\dot{W}_2^2(\Omega)} + \|u\|_{W_2^\tau(\Omega_2)}\}$$

> for $\kappa = 0, 1$ holds true.

In order to get this 'optimal' local convergence a special super-approximability property of finite elements is used. In addition certain global shift properties of the operator $-\Delta$ are needed.

The construction of an operator-algebra consisting of integral and differential operators leads to the concept of pseudo-differential operators. The counterpart of (0.2) resp. (0.4) regarding the application of the Ritz method is Gårding's inequality for strong elliptic pseudo-differential operators (see Schatz [16]). For such equations the Ritz method is almost best approximating with respect to the corresponding 'energy-norm'. The Examples 1 and 2 are model problems with strong elliptic pseudo-differential operators of order 2α and the 'energy-norm' $\|\cdot\|_\alpha$ with $\alpha = 1$ resp. $\alpha = -1/2$.

In the present paper we will derive the 'optimal' local convergence of the Ritz method for strong elliptic pseudo-differential operators. We emphasize that our treatment covers simultaneously operators of positive and negative order.

We will use the local shift properties of elliptic pseudo-differential operators P (see Treves [18] p. 42):

> Let w be the solution of an elliptic pseudo-differential equation $Pw=f$. If f is in $C^{\infty}(\Omega')$ for some open domain Ω' then also $w \in C^{\infty}(\Omega')$.

1. Global Error Estimates

Let $\{H_\beta \mid \beta \in \mathbf{R}\}$ be a Hilbert-scale with the special assumption:

For $\beta = m \in \mathbf{N}_0$ (the set of all nonnegative integers) the spaces

$$H_m \subseteq W_2^m(\Omega) \tag{1.1}$$

are subspaces of the Sobolev-space $W_2^m(\Omega)$ with

$$\Omega = \Sigma \quad \text{or} \quad \Omega = \partial\Sigma \tag{1.2}$$

and Σ being a bounded domain with boundary $\partial\Sigma$ sufficiently smooth.

$(\cdot, \cdot)_\beta$, $\|\cdot\|_\beta$ will denote the inner product respectively the norm in H_β. In case of $\beta = 0$ we skip the subscript.

We assume that the operator A (0.1) has the following properties:

1) There is an $\alpha \in \mathbf{R}$ such that
   i) The mapping $A: H_{\beta+2\alpha} \to H_\beta$ is an isomorphism for $\beta \in \mathbf{R}$, i.e.
   $$c^{-1}\|u\|_{\beta+2\alpha} \leq \|Au\|_\beta \leq c\|u\|_{\beta+2\alpha}$$
   with some constant c.
   ii) A is positive definite in H_α, i.e.
   $$(u, Au) \geq \underline{c}\|u\|_\alpha^2 \tag{1.3}$$
   with $\underline{c} > 0$.
2) A is self-adjoint in $H = H_0$, i.e.
   $$(Au, v) = (u, Av)$$
   for $u, v \in D(A)$.

By

$$a(u, v) = (Au, v) \qquad \text{for} \quad u, v \in D(A) \tag{1.4}$$

an inner product is defined.

LEMMA 1.1. *There is a constant c (depending on β) such that*

$$c^{-1}\|u\|_\beta \leq \sup_{\substack{v\in H_{\beta*}\\ v\neq 0}} \frac{a(u,v)}{\|v\|_{\beta*}} \leq c\|u\|_\beta \quad \text{for} \quad u\in H_\beta \tag{1.5}$$

with $\beta^* := 2\alpha - \beta$.

REMARK 1.2. In the following we will denote with c numerical constants which may differ at different places.

Proof. The right part of the inequality (1.5) is a direct consequence of Schwarz' inequality and (1.3i).

By the standard inequality in Hilbert-scales we have

$$\|u\|_\beta \leq c \sup_{\substack{w\in H_{-\beta}\\ w\neq 0}} \frac{(u,w)}{\|w\|_{-\beta}}. \tag{1.6}$$

In order to show the left inequality we define for $w\in H_{-\beta}$ an auxiliary function v by $Av=w$. On the one hand it is $(u, w) = a(u, v)$ and on the other hand

$$\|v\|_{\beta*} \leq c\|Av\|_{\beta*-2\alpha} = c\|w\|_{-\beta}. \tag{1.7}$$

♯

Since we will consider only 'approximation-spaces' which are contained in $L_2 = H$ we impose the following regularity in order to apply the Ritz method:

$$S_h \subseteq H_a \quad \text{with} \quad a := \max\{0, \alpha\}. \tag{1.8}$$

In our analysis we will need the regularity

$$S_h \subseteq H_s \quad \text{with} \quad s := \begin{cases} 2a, & 2a\in N_0 \\ [2a]+1, & 2a\notin N_0 \end{cases} \tag{1.9}$$

which is an additional assumption only in case of $\alpha > 0$. For any linear bounded operator $B\colon H_\gamma \to H_\beta$ we introduce the norm

$$\|B\|_{\beta\cdot\gamma} := \sup_{\substack{u\in H_\gamma\\ u\neq 0}} \frac{\|Bu\|_\beta}{\|u\|_\gamma}. \tag{1.10}$$

THEOREM 1.3. *The Ritz operator* $R_h\colon H_\gamma \to S_h \subseteq H_\beta$ *defined by* (0.6) *admits for* $\beta, \gamma \in [s^*, s]$ *with* $s^* := 2\alpha - s$ *the estimate*

$$c^{-1}\|R_h\|_{\beta\cdot\gamma} \leq \|R_h\|_{\gamma^*\cdot\beta^*} \leq c\|R_h\|_{\beta\cdot\gamma}. \tag{1.11}$$

Proof. Because of $(\beta^*)^* = \beta$ and $(\gamma^*)^* = \gamma$ it is sufficient to show one of the inequalities. Using (0.8) and (1.5) we get

$$\begin{aligned} \|R_h\|_{\beta\cdot\gamma} &= \sup_{\substack{u\in H_\gamma\\ u\neq 0}} \frac{\|R_h u\|_\beta}{\|u\|_\gamma} \leq c \sup_{\substack{u\in H_\gamma\\ u\neq 0}} \sup_{\substack{v\in H_{\beta*}\\ v\neq 0}} \frac{a(R_h u, v)}{\|v\|_{\beta*}\|u\|_\gamma} \\ &\leq c \sup_{\substack{v\in H_{\beta*}\\ v\neq 0}} \sup_{\substack{u\in H_\gamma\\ u\neq 0}} \frac{a(u, R_h v)}{\|u\|_\gamma \|v\|_{\beta*}} \leq c \sup_{\substack{v\in H_{\beta*}\\ v\neq 0}} \frac{\|R_h v\|_{\gamma*}}{\|v\|_{\beta*}} \\ &= c\|R_h\|_{\gamma^*\cdot\beta^*}. \end{aligned} \tag{1.12}$$

♯

By

$$N_\beta(\psi):=\sup_{\chi\in S_h}\frac{a(\psi,\chi)}{\|\chi\|_{\beta*}}\qquad\text{for}\quad\psi\in S_h \tag{1.13}$$

a norm is defined in S_h. For S_h finite dimensional this new norm is equivalent to the β-norm. Obviously we have

$$N_\beta(\psi)\le c\|\psi\|_\beta\,. \tag{1.14}$$

We introduce κ_h by

$$\kappa_h:=\sup\{\|\psi\|_\beta\,|\,\psi\in S_h,\ N_\beta(\psi)=1\} \tag{1.15}$$

and show

THEOREM 1.4. *The following assertions are equivalent*:

$$\begin{aligned}
&\text{i)}\quad \|R_h\|_{\beta\cdot\beta}\le c\,,\\
&\text{ii)}\quad \tau_h:=\kappa_h^{-1}\ge\underline{\tau}>0\ (\textit{with } \underline{\tau} \textit{ independent of } h)\,,\\
&\text{iii)}\quad \inf_{\psi\in S_h}\sup_{\chi\in S_h}\frac{a(\psi,\chi)}{\|\chi\|_{\beta*}\|\psi\|_\beta}\ge\underline{\tau}>0\qquad \textit{for each } \beta\in[s^*,s].
\end{aligned} \tag{1.16}$$

REMARK 1.5. Theorem 1.4 may be considered as a generalization of the Polskii condition (see Polskii [15]). We notice that (1.16i) holds if and only if the Ritz method is almost best approximating in the β-norm (see Alexits [1]). With respect to (1.16iii) we refer to Aziz-Kellog [3].

Proof. (0.8) and Theorem 1.3 give for $\psi\in S_h$

$$\begin{aligned}
\|\psi\|_\beta&\le c\sup_{\substack{v\in H_{\beta*}\\ v\ne0}}\frac{a(\psi,v)}{\|v\|_{\beta*}}=c\sup_{\substack{v\in H_{\beta*}\\ v\ne0}}\frac{a(\psi,R_hv)}{\|v\|_{\beta*}}\\
&=c\sup_{\substack{v\in H_{\beta*}\\ R_hv\ne0}}\frac{a(\psi,R_hv)}{\|R_hv\|_{\beta*}}\cdot\frac{\|R_hv\|_{\beta*}}{\|v\|_{\beta*}}\\
&\le cN_\beta(\psi)\|R_h\|_{\beta*\cdot\beta*}
\end{aligned} \tag{1.17}$$

which shows

$$\kappa_h\le c\|R_h\|_{\beta*\cdot\beta*}\le c\|R_h\|_{\beta\cdot\beta}\,. \tag{1.18}$$

On the other hand we find with (0.8) for $u_h=R_hu\in S_h$

$$\begin{aligned}
\|u_h\|_\beta&\le\kappa_hN_\beta(u_h)\\
&=\kappa_h\sup_{\chi\in S_h}\frac{a(u,\chi)}{\|\chi\|_{\beta*}}\\
&\le c\kappa_h\|u\|_\beta
\end{aligned} \tag{1.19}$$

and therefore

(1.20) $$\|R_h\|_{\beta \cdot \beta} \leq c\kappa_h .$$

Thus the equivalence of (1.16i) and (1.16ii) is shown. The equivalence of (1.16ii) and (1.16iii) is obvious. #

We will use certain approximation properties of the spaces S_h:

DEFINITION 1.6. We use the notation $S_h = S_h^{k \cdot t}$ with $k < t$ if the following statements hold true:

i) $S_h \subseteq H_k$,

(1.21) ii) $\inf_{\chi \in S_h} \|v - \chi\|_k \leq ch^{t-k} \|v\|_t$ for $v \in H_t$,

iii) $\|\chi\|_k \leq ch^{-(k-k')} \|\chi\|_{k'}$ for $\chi \in S_h$

for $k' < k$.

REMARK 1.7. In the one dimensional case the trigonometric polynomials of degree n share these properties with $h = n^{-1}$ for any (k, t).

REMARK 1.8. If S_h is spanned by piecewise polynomial functions subject to regular subdivision of Ω then the conditions of Definition 1.6 hold true if the elements of S_h are global in C^{k-1} and the degree of the polynomials is at least $t-1$.

The Bramble-Scott-Lemma (see [5]) gives

LEMMA 1.9. *Let $\underline{\beta}$, $\bar{\beta}$ be fixed with $\underline{\beta} < \bar{\beta} \leq k$. To any $v \in H_\tau$ with $\bar{\beta} \leq \tau \leq t$ there exists a $\chi \in S_h^{k \cdot t}$ with*

(1.22) $$\|v - \chi\|_\beta \leq ch^{\tau - \beta} \|v\|_\tau$$

simultaneously for $\beta \in [\underline{\beta}, \bar{\beta}]$.

With the help of the logarithmic convexity of the norms in a Hilbert-scale the inverse properties

(1.23) $$\|\chi\|_\beta \leq ch^{-(\beta - \beta')} \|\chi\|_{\beta'} \quad \text{for} \quad \chi \in S_h$$

are valid for any pair (β', β) with $\beta' \leq \beta \leq k$ and $\beta' \leq k'$. The standard error estimates in our setting are —see the assumptions (1.3)—

THEOREM 1.10. *Let $u_h \in S_h^{k \cdot t} \subseteq H_s$ be the Ritz approximation on a function $u \in H_\tau$ with $\beta \leq \tau \leq t$. Then the error estimate*

(1.24) $$\|u - u_h\|_\beta \leq ch^{\tau - \beta} \|u\|_\tau$$

holds for $\beta \in [t^*, k]$ with $t^* = 2\alpha - t$.

REMARK 1.11. Up to now our only assumptions on S_h are the approximation properties of Definition 1.6. As a consequence the error estimate (1.24) is valid for conforming finite element methods, boundary finite element methods, spectral and pseudo-spectral finite element methods.

2. Local Error Estimates

In addition to (1.3) we assume that the following properties are valid:

i) Let $\Omega' \subseteq \Omega$ and $Av \in C^\infty(\Omega') = \bigcap_{\beta \in \mathbf{R}} H_\beta(\Omega')$, then $v \in C^\infty(\Omega')$.

ii) Let ρ and σ be cut-off functions, i.e. $\rho, \sigma \in C_0^\infty(\Omega)$, with $\operatorname{supp}(\rho) \cap \operatorname{supp}(\sigma) = \emptyset$. To any pair β, $m \in \mathbf{R}$ there is a constant c with

$$\|\rho A \sigma v\|_{\beta+m} \leq c\|v\|_\beta \qquad \text{for} \quad v \in H_\beta . \tag{2.1}$$

iii) The operator $A_2 := \omega A - A\omega$ with $\omega \in C^\infty(\Omega)$ is of order $2\alpha - 1$, i.e.

$$\|A_2 v\|_\beta \leq c\|v\|_{\beta+2\alpha-1}$$

for any $\beta \in \mathbf{R}$.

The following local approximation properties of the spaces $S_h = S_h^{k,t}(\Omega)$ are typical for finite elements with κ-regular subdivision (see Nitsche-Schatz [13] and the literature cited):

E.1 [LOCAL APPROXIMABILITY]. *Let $v \in H_\tau(\Omega)$ with $\tau \leq t$ be fixed and let $\Omega_1 :=$ $\operatorname{supp}(v) \Subset \Omega$ be contained properly in Ω. There exists a second domain Ω' with $\Omega_1 \Subset \Omega' \subseteq \Omega$ and $h_0 > 0$ depending on* $\operatorname{dist}(\Omega_1, \Omega')$ *such that for $h \leq h_0$ there exists a $\chi \in S_h$ with*

$$\begin{aligned} &\text{i)} \quad \operatorname{supp}(\chi) \subseteq \Omega', \\ &\text{ii)} \quad \operatorname{dist}(\Omega_1, \operatorname{supp}(\chi)) \leq ch, \\ &\text{iii)} \quad \|v - \chi\|_{\lambda, \Omega'} \leq ch^{\tau-\lambda}\|v\|_{\tau, \Omega_1} \\ &\qquad \textit{for integer } \lambda \textit{ with } 0 \leq \lambda \leq k \textit{ and } \lambda < \tau. \end{aligned} \tag{2.2}$$

The constant c is independent of v and h.

E.2 [SUPER-APPROXIMABILITY]. *Let $\omega \in C^\infty(\Omega)$ be fixed such that the inclusions $\Omega_1 := \operatorname{supp}(\omega) \Subset \Omega' \subseteq \Omega$ hold. There is an $h_0 > 0$ (depending on* $\operatorname{dist}(\Omega_1, \Omega')$*) such that for $h \leq h_0$ the function $\omega\varphi$ with $\varphi \in S_h$ arbitrary can be approximated by a function $\chi \in S_h$ such that*

$$\begin{aligned} &\text{i)} \quad \operatorname{supp}(\chi) \subseteq \Omega', \\ &\text{ii)} \quad \|\omega\varphi - \chi\|_{\lambda, \Omega} \leq ch^{k+1-\lambda}\|\varphi\|_{k, \Omega'} \quad \textit{for } 0 \leq \lambda \leq k. \end{aligned} \tag{2.3}$$

The constant c will depend only on ω and its derivatives up to order k as well as on the distance $\operatorname{dist}(\Omega_1, \Omega')$.

A consequence of E.1 is (see Nitsche-Schatz [13]):

LEMMA 2.1. *Let $v \in H_\tau(\Omega_2)$ with $\tau \leq t$ and $\Omega_1 \Subset \Omega_2 \subseteq \Omega$ be given. There exists a $\chi \in S_h$ such that*

$$\text{(2.4)}\quad \begin{array}{l}\text{i)}\ \ \operatorname{supp}(\chi)\subseteq\Omega_2,\\ \text{ii)}\ \ \|v-\chi\|_{\lambda\cdot\Omega_1}\le ch^{\tau-\lambda}\|v\|_{\tau\cdot\Omega_2}\quad \textit{for } 0\le\lambda\le k \textit{ and } \lambda<\tau.\end{array}$$

The super-approximability (2.3) is restricted to integer λ. For 'negative' norms we will show

LEMMA 2.2. *Let ω, Ω_1, Ω, and h_0 be as in E.2. Further let P_h be the orthogonal projection of H onto S_h. Then for $\varphi\in S_h$ arbitrary the estimate*

$$\text{(2.5)}\qquad \|\omega\varphi-P_h(\omega\varphi)\|_{-l}\le ch\|\varphi\|_{-l}$$

holds true for real l with $0\le l\le t$ and c independent of φ and h.

Proof. Because of the characterization

$$\text{(2.6)}\qquad \|\omega\varphi-P_h(\omega\varphi)\|_{-l}=\sup_{\substack{v\in H_l\\ v\neq 0}}\frac{(\omega\varphi-P_h(\omega\varphi),v)}{\|v\|_l}$$

and the fact that P_h is the orthogonal projection we get with $\chi\in S_h$ arbitrary

$$\text{(2.7)}\qquad \|\omega\varphi-P_h(\omega\varphi)\|_{-l}=\sup_{\substack{v\in H_l\\ v\neq 0}}\frac{(\omega\varphi-P_h(\omega\varphi),v-\chi)}{\|v\|_l}.$$

We choose $\chi\in S_h$ corresponding to the approximation properties (1.21ii) of the spaces S_h and get with (2.3) and (1.23)

$$\text{(2.8)}\qquad \begin{aligned}\|\omega\varphi-P_h(\omega\varphi)\|_{-l}&\le ch^l\|\omega\varphi-P_h(\omega\varphi)\|_0\\ &\le ch^{l+1}\|\varphi\|_0\\ &\le ch\|\varphi\|_{-l}.\end{aligned}$$

♯

In the proof of Lemma 2.4 below we will apply Lemma 2.2 in case of $\alpha<0$ with $l=2|\alpha|$. In case of $\alpha>0$ we consider for simplicity only α with $\alpha\in N$. In order to do this the superscripts k and t characterizing the spaces $S_h=S_h^{k\cdot t}(\Omega)$ are subject to

$$\text{(2.9)}\qquad \begin{array}{ll}0\le k=2\alpha<t & \text{for}\quad \alpha>0\\ 0=k<2|\alpha|\le t & \text{for}\quad \alpha<0.\end{array}$$

The main result of our paper is

THEOREM 2.3. *Let u be the solution of* (0.1) *and assume the regularity $u\in H_a(\Omega)\cap H_\tau(\Omega_2)$ with $a<\tau\le t$ and $\Omega_2\subseteq\Omega$. Further let Ω_1 be a second domain with $\Omega_1\Subset\Omega_2$ and h_0 chosen properly. The error $E:=u-u_h$ between u and the Ritz approximation u_h* (0.6) *admits for $h\le h_0$ the local estimate ($t^*=2\alpha-t$)*

$$\text{(2.10)}\qquad \|E\|_{0\cdot\Omega_1}\le c\left\{h^\tau(\|u\|_{\tau\cdot\Omega_2}+\|u\|_{a\cdot\Omega})+\|E\|_{t^*}+h^{t-a}\inf_{\chi\in S_h}\|u-\chi\|_a\right\}.$$

In proving the theorem the essential step is

LEMMA 2.4. *Let u, τ, Ω_1 etc. be as in Theorem 2.3 and let Ω_2' be chosen such that $\Omega_1 \Subset \Omega_2' \Subset \Omega_2$. Then*

$$\|E\|_{0 \cdot \Omega_1} \leq c \left\{ h^\tau \|u\|_{\tau \cdot \Omega_2} + \|E\|_{t^*} + h^{t-a} \inf_{\chi \in S_h} \|u - \chi\|_a \right\} + ch\|E\|_{0 \cdot \Omega_2'} . \tag{2.11}$$

Before proving the lemma we show that Theorem 2.3 is a consequence: Let additional domains be chosen such that

$$\Omega_1 \Subset \Omega_2' =: \Omega_1'' \Subset \cdots\cdots \Subset \Omega_{[\tau]+1}'' := \Omega_2 , \tag{2.12}$$

then we apply Lemma 2.4 successively with Ω_1, Ω_2' replaced by Ω_i'', Ω_{i+1}'', which finally gives the inequality stated in Theorem 2.3 (since $\|E\|_{0 \cdot \Omega_2} \leq \|E\|_{0 \cdot \Omega} \leq c\|u\|_{0 \cdot \Omega} \leq c\|u\|_{a \cdot \Omega}$).

In order to prove Lemma 2.4 we consider similarly additional domains Ω_i'' as above in the following way

$$\Omega_1 =: \Omega_1'' \Subset \cdots\cdots \Subset \Omega_9'' := \Omega_2' . \tag{2.13}$$

Let $\omega_i \in C_0^\infty(\Omega)$ $(1 \leq i \leq 8)$ be cut-off functions with respect to Ω_i'' and Ω_{i+1}'' such that

$$\begin{aligned} &\text{i)} \quad \omega_i \equiv 1 \text{ in } \Omega_i'', \\ &\text{ii)} \quad \operatorname{supp}(\omega_i) \Subset \Omega_{i+1}'', \\ &\text{iii)} \quad 0 \leq \omega_i \leq 1, \end{aligned} \tag{2.14}$$

and put

$$\hat{\omega}_i := 1 - \omega_i . \tag{2.15}$$

With the help of an appropriate approximation $\Psi \in S_h$ on u we use the splitting

$$\begin{aligned} E &= (u - \Psi) - (u_h - \Psi) \\ &=: \theta - \Phi . \end{aligned} \tag{2.16}$$

Because of Lemma 2.1 we may choose Ψ such that

$$\|\theta\|_{0 \cdot \Omega_2'} \leq ch^\tau \|u\|_{\tau \cdot \Omega_2} . \tag{2.17}$$

With the help of ω_1 we may estimate

$$\|E\|_{0 \cdot \Omega_1}^2 \leq \|E\|_{\omega_1}^2 := (\omega_1 E, E) . \tag{2.18}$$

Let the auxiliary function w be defined by

$$Aw = \omega_1 E \in H . \tag{2.19}$$

Because of (1.3i) and (2.1i) we have the regularity

$$w \in H_{2\alpha}(\Omega) \cap C^\infty(\Omega - \Omega_2'') . \tag{2.20}$$

We denote by $w_h := R_h w \in S_h$ the Ritz approximation on w. For the error

$$e := w - w_h \tag{2.21}$$

we have the defining relation

$$(2.22)\qquad (Ae,\chi)=0 \qquad \text{for all} \quad \chi\in S_h\,.$$

Analogue to (2.16) we use the splitting

$$(2.23)\qquad \begin{aligned} e&=(w-\psi)-(w_h-\psi) \qquad \text{with} \quad \psi\in S_h\\ &=:\varepsilon-\varphi\,. \end{aligned}$$

The choice of ψ is crucial. According to the representation $w=\omega_2 w+\hat{\omega}_2 w$ we use $\psi=\psi_2+\hat{\psi}_2\in S_h$ with ψ_2, $\hat{\psi}_2$ defined by

(2.24)
i) $\psi_2\in S_h$ is an approximation on $\omega_2 w$ with $\operatorname{supp}(\psi_2)\subseteq\Omega_4''$ according to the local approximability E.1,
ii) $\hat{\psi}_2\in S_h$ is an approximation on $\hat{\omega}_2 w\in C^\infty(\Omega)$ according to Lemma 2.1.

For—see (2.23)—

$$(2.25)\qquad \begin{aligned} \varepsilon&=(w_2 w-\psi_2)+(\hat{\omega}_2 w-\hat{\psi}_2)\\ &=:\varepsilon_2+\hat{\varepsilon}_2 \end{aligned}$$

we get—see (2.19), (2.20)—

$$(2.26)\qquad \begin{aligned} &\text{i)}\quad \operatorname{supp}(\varepsilon_2)\subseteq\Omega_4'',\ \|\varepsilon_2\|_{2\alpha}\le c\|E\|_{\omega_1},\\ &\text{ii)}\quad \|\hat{\varepsilon}_2\|_a\le ch^{t-a}\|E\|_{\omega_1}. \end{aligned}$$

Now we turn to the

Proof of Lemma 2.4. Because of (0.8) we get from (2.19) with any $\chi\in S_h$

$$(2.27)\qquad \|E\|^2_{0\cdot\Omega_1}\le\|E\|^2_{\omega_1}=(AE,w)=(AE,w-\chi)\,.$$

The special choice $\chi:=\psi\in S_h$ —see (2.23)—leads to

$$(2.28)\qquad \|E\|^2_{\omega_1}=(E,A\varepsilon)$$

which we split as follows

$$(2.29)\qquad \begin{aligned} \|E\|^2_{\omega_1}&=(E,\hat{\omega}_5 A\varepsilon)+(E,\omega_5 A\varepsilon)\\ &=:T_1+T_2\,. \end{aligned}$$

Using Theorem 1.10, (2.1ii), (2.26), and the fact that ω_1 and $\hat{\omega}_{i+1}$ have disjoined supports we come to the following sequence of inequalities for the first term T_1 on the right hand side in (2.29)

$$(2.30)\qquad \begin{aligned} |T_1|&=|(E,\hat{\omega}_5 A\omega_4\varepsilon)+(E,\hat{\omega}_5 A\hat{\omega}_4\varepsilon)|\\ &\le c\{\|E\|_{t^*}\|\hat{\omega}_5 A\omega_4\varepsilon\|_{-t^*}+\|\hat{\omega}_5 E\|_a\|A\hat{\omega}_4\hat{\varepsilon}_2\|_{-a}\}\\ &\le c\{\|E\|_{t^*}\|\varepsilon\|_{2\alpha}+\|\hat{\omega}_5 E\|_a\|\hat{\omega}_4\hat{\varepsilon}_2\|_{-a+2\alpha}\} \end{aligned}$$

$$\leq c\{\|E\|_{t^*}\|w\|_{2\alpha}+\|E\|_a\|\hat{\varepsilon}_2\|_a\}$$

$$\leq c\|E\|_{\omega_1}\left\{\|E\|_{t^*}+h^{t-a}\inf_{\chi\in S_h}\|u-\chi\|_a\right\}.$$

In order to estimate the second term T_2 we use the identity

$$T_2=(\omega_5 E, A\varepsilon)$$
$$=(\omega_5 E, Ae)+(E, \omega_5 A\varphi)$$
$$=(\omega_5\theta, Ae)-(\omega_5\Phi, Ae)+(E, (\omega_5 A-A\omega_5)\varphi)+(E, A\omega_5\varphi)\,. \tag{2.31}$$

Because of the defining relations for E and e we can rewrite T_2 with $\xi, \eta\in S_h$ arbitrary

$$T_2=(\omega_5\theta, Ae)-(\omega_5\Phi-\xi, Ae)+(E, A_2\varphi)+(AE, \omega_5\varphi-\eta)\,. \tag{2.32}$$

Here A_2 is defined by $A_2:=\omega_5 A-A\omega_5$. We choose $\xi\in S_h$ such that the super-approximability property (2.3) with ω, φ replaced by ω_5, Φ is fulfilled. With the help of (2.16), (2.17), and Theorem 1.10 we find the bound needed for the first two terms in (2.32)

$$|(\omega_5\theta, Ae)-(\omega_5\Phi-\xi, Ae)|\leq\|Ae\|_0\{\|\theta\|_{\omega_5}+\|\omega_5\Phi-\xi\|_0\}$$
$$\leq c\|E\|_{\omega_1}\{h^\tau\|u\|_{\tau\cdot\Omega_2}+h\|E\|_{0\cdot\Omega_2'}\}\,. \tag{2.33}$$

The third term in (2.32) can be estimated with the help of (2.1ii) and (2.1iii)

$$|(E, A_2\varphi)|=|(E, \omega_6 A_2\varphi)+(E, \hat{\omega}_6 A_2\varphi)|$$
$$=|(E, \omega_6 A_2\varphi)-(E, \hat{\omega}_6 A\omega_5\varphi)|$$
$$\leq\|E\|_{\omega_6}\|A_2\varphi\|_0+\|E\|_{t^*}\|\hat{\omega}_6 A\omega_5\varphi\|_{-t^*}$$
$$\leq c\{\|E\|_{\omega_6}\|\varphi\|_{2\alpha-1}+\|E\|_{t^*}\|\varphi\|_{2\alpha}\}$$
$$\leq c\|w\|_{2\alpha}\{h\|E\|_{\omega_6}+\|E\|_{t^*}\}$$
$$\leq c\|E\|_{\omega_1}\{h\|E\|_{0\cdot\Omega_2'}+\|E\|_{t^*}\}\,. \tag{2.34}$$

In order to estimate the fourth term in (2.32) we choose

$$\eta:=P_h(\omega_5\varphi)\,. \tag{2.35}$$

Since the L_2-projection has 'optimal' local convergence (see Nitsche-Schatz [13]) we get

$$\|\hat{\omega}_7(\omega_5\varphi-\eta)\|_a\leq ch^{t-a}\|E\|_{\omega_1}\,. \tag{2.36}$$

Using (1.3i), (2.3) resp. (2.5), (2.36), and Theorem 1.10 we come to the final sequence of inequalities

$$
\begin{aligned}
&|(AE, \omega_5\varphi-\eta)| \\
&\quad =|(E, A(\omega_5\varphi-\eta))| \\
&\quad =|(E, \omega_8 A(\omega_5\varphi-\eta))+(E, \hat{\omega}_8 A\omega_7(\omega_5\varphi-\eta))+(E, \hat{\omega}_8 A\hat{\omega}_7(\omega_5\varphi-\eta))| \\
&\quad \le \|E\|_{\omega_8}\|A(\omega_5\varphi-\eta)\|_0+\|E\|_{t^*}\|\hat{\omega}_8 A\omega_7(\omega_5\varphi-\eta)\|_{-t^*}+\|E\|_a\|A\hat{\omega}_7\eta\|_{-a} \\
&\quad \le c\{\|E\|_{\omega_8}\|\omega_5\varphi-\eta\|_{2\alpha}+\|E\|_{t^*}\|\omega_5\varphi-\eta\|_{2\alpha}+\|E\|_a\|\hat{\omega}_7\eta\|_a\} \\
&\quad \le c\{(\|E\|_{\omega_8}+\|E\|_{t^*})h\|\varphi\|_{2\alpha}+h^{t-a}\|E\|_a\|E\|_{\omega_1}\} \\
&\quad \le c\|E\|_{\omega_1}\left\{h\|E\|_{0\cdot\Omega_2'}+\|E\|_{t^*}+h^{t-a}\inf_{\chi\in S_h}\|u-\chi\|_a\right\}.
\end{aligned}
\tag{2.37}
$$

This completes the proof of Lemma 2.4. ♯

We will use Example 2 to give an illustration of Theorem 2.3: Because of $\alpha=-\frac{1}{2}$ and therefore $a=\max\{0, \alpha\}=0$ condition (2.9) for the superscripts k and t characterizing the spaces $S_h^{k\cdot t}$ leads to

$$0=k<t\,. \tag{2.38}$$

For a (uniform) κ-regular subdivision $\gamma=\gamma_h$ of the interval $I:=(-\pi, \pi)$ we consider piecewise linear, periodic splines

$$S_h:=S_\gamma^{0\cdot 2}(I)\subseteq L_2^\sharp(I)\,. \tag{2.39}$$

Let $u\in H_\sigma^\sharp(I)\cap H_\tau(I_2)$ with $I_2\subseteq I$ and $0\le\sigma<\tau\le 2$ be the periodic solution of (0.14). Then the third and fourth term on the right hand side in (2.10) give ($t^*=2\alpha-t=-3$)

$$\|E\|_{t^*}\le ch^{\sigma+3}\|u\|_\sigma$$

(2.40) resp.

$$h^{t-a}\inf_{\chi\in S_h}\|u-\chi\|_a\le h^2\|u\|_0\,.$$

Therefore we only need the global regularity assumption ($\sigma=0$) $u\in L_2^\sharp(I)$ in order to get the 'optimal' local error estimate

$$\|u-u_h\|_{0\cdot I_1}\le ch^\tau\{\|u\|_{\tau\cdot I_2}+\|u\|_{0\cdot I}\} \tag{2.41}$$

for $I_1\Subset I_2$.

References

[1] G. Alexits, Einige Beiträge zur Approximationstheorie. Acta Sci. Math., **16** (1965), 212–224.

[2] J. P. Aubin, Approximation des espaces de distributions et des opérateurs différentiels. Bull. Soc. Math. France, Suppl. **12** (1967).

[3] A. K. Aziz and R. B. Kellog, Finite element analysis of scattering problem. Math. Comp., **37** (1981), 261–272.

[4] I. Babuska, Error bounds for finite element method. Numer. Math., **16** (1971), 322–333.

[5] J. H. Bramble and R. Scott, Simultaneous approximation in scales of Banach spaces. Math. Comp., **32** (1978), 947–954.

[6] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[7] H. P. Helfrich, Simultaneous approximation in negative norms of arbitrary order. RAIRO Anal. Numér., **15** (1981), 231–235.

[8] G. C. Hsiao and W. L. Wendland, A finite element method for some integral equations of the first kind. J. Math. Anal. Appl., **58** (1977), 449–481.

[9] J. J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators. Comm. Pure Appl. Math., **18** (1965), 269–305.

[10] S. G. Krein and I. Petunin, Scales of Banach spaces. Russian Math. Surveys, **21** (1966), 85–159.

[11] J. Nitsche, Ein Kriterium für die Quasi-Optimalität des Ritz'schen Verfahrens. Numer. Math., **11** (1968), 346–348.

[12] J. Nitsche, Zur Konvergenz von Näherungsverfahren bezüglich verschiedener Normen. Numer. Math., **15** (1970), 224–228.

[13] J. Nitsche and A. H. Schatz, On local approximation properties of L_2-projections on spline subspaces. Applicable Anal., **2** (1972), 161–168.

[14] J. Nitsche and A. H. Schatz, Interior estimates for Ritz-Galerkin methods. Math. Comp., **28** (1974), 937–958.

[15] N. I. Polskii, On a general scheme of application of approximation methods. Dokl. Akad. Nauk, **111** (1956), 1181–1184.

[16] A. Schatz, An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. Math. Comp., **28** (1974), 959–962.

[17] E. Stephan and W. L. Wendland, Remarks to Galerkin and least squares methods with finite elements for general elliptic problems. Manuscripta Geodaetica, **1** (1976), 93–123.

[18] F. Treves, Introduction to Pseudodifferential and Fourier Integral Operators I, II. Plenum Press, New York-London, 1980.

[19] G. Vainikko, On the question of convergence of Galerkin's method. Tartu Riikl. Ül. Toimetised, **177** (1965), 148–152.

[20] M. Zlamal, Curved elements in the finite element method. Part I: SIAM J. Numer. Anal., **10** (1973), 229–240, Part II: SIAM J. Numer. Anal., **11** (1974), 347–362.