Direct Proofs of Some Unusual Shift–Theorems

Joachim A. Nitsche

Dedicated to Prof. Dr. Jacques L. Lions
on His 60th Birthday

Abstract: For the Poisson equation, the plate equation and the two dimensional Stokes' equations direct proofs are given concerning the regularity of the solution in dependence of a reduced regularity of the right hand sides based on standard estimates for the Newtonian potential.

0. Let us consider a boundary value problem of the type $\mathcal{A}u = f$ in function spaces $u \in \mathcal{H}_1$ and $f \in \mathcal{H}_2$. Usually for a given space $\mathcal{H}_1$ the range $\mathcal{H}_2 = \mathcal{R}(\mathcal{A}) = \{f \mid f = \mathcal{A}u \wedge u \in \mathcal{H}_1\}$ is obvious to describe such that the mapping $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded. Typical examples for $\mathcal{A} = -\Delta$ with Δ being the Laplacian are $\mathcal{H}_1 = H_k(\Omega)$ and $\mathcal{H}_2 = H_{k-2}(\Omega)$ respective $\mathcal{H}_1 = C_{k,\lambda}(\overline{\Omega})$ and $\mathcal{H}_2 = C_{k-2,\lambda}(\overline{\Omega})$ for $k \geq 2$. Here H_k respective $C_{k,\lambda}$ denote the usual Sobolev respective Hölder spaces. The concern of shift–theorems is the reciprocal problem, i.e. to characterize function spaces $\mathcal{H}_1$, $\mathcal{H}_2$ such that the mapping $\mathcal{A}^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ is bounded. In the above mentioned examples $\mathcal{A}^{-1}$ is bounded provided the boundary $\partial\Omega$ is sufficiently regular and e.g. the boundary values are zero. It is easy to see that the mentioned shift theorems may be extended to $k = 1$ in the case of Sobolev spaces. Then $H_{-1}(\Omega)$ may be considered as the dual of $H_1(\Omega)$ with respect to $L_2(\Omega)$ or equivalently (in the distributional sense) as the space of functions given by the divergence of vector-valued functions with components in $L_2(\Omega)$ equipped with the appropriate L_2-norm. If the space $C_{-1,\lambda}(\overline{\Omega})$ is defined in the latter way the corresponding shift theorem is also valid.

This result is contained in the famous paper of Agmon-Douglas-Nirenberg (1959). We refer also to Morrey (1966), {Theorem 5.5.5, p. 156} and concerning interior estimates to Giaquinta (1983), {Theorem 2.2, p. 84}.

In the literature cited the full strength of the theory of elliptic equations is used. The aim of this paper is to give direct proofs of this shift theorem and corresponding generalizations to the biharmonic operator as well as to the Stokes' equations in two dimensions. The proofs are based on standard estimates for the Newtonian potential. The underlying idea can be found in Schulz (1981). In this way the proofs presented here may be considered as a consequent continuation of the work of Schulz.

Quite often we will refer to the standard book of Gilbarg-Trudinger (1977), the references are indicated by {GT, p. ...}. Of course the notations used are those of the book cited. The partial derivatives "$\partial z/\partial x_i$", "$\partial^2 z/\partial x_i \partial x_j$" etc. will be abbreviated by "$z_{|i}$", "$z_{|ij}$" etc. The summation convention is not used, whenever a sum occurs for instance with respect to "i" we will indicate this by $\sum_{(i)}$.

1. In this section we consider the boundary value problem

$$(1.1)\qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega \quad , \\ u &= 0 \quad \text{on } \partial\Omega \quad . \end{aligned}$$

Here $\Omega \subset \mathbb{R}^n$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. The Sobolev-space theory gives the shift theorem:

Theorem (S.1): Assume the right hand side in (1.1) has the regularity $f \in H_k = H_k(\Omega)$ with $k \geq 0$. Then the unique (generalized) solution of the boundary value problem (1.1) has the regularity $u \in H_{k+2}$ and the a priori estimate holds true:

$$(1.2)\qquad \| u \|_{H_{k+2}} \leq c \| f \|_{H_k} \quad .$$

Remark: c will denote a generic constant which may differ at different places. In (1.2) c depends only on k and $\partial\Omega$. If necessary we will write e. g. $c = c(k,r)$ in order to indicate the dependence of c upon the two other constants k and r.

Parallel to above the Schauder theory gives the shift theorem:

Theorem (S.2): Assume the right hand side in (1.1) has the regularity $f \in C_{k.\lambda} = C_{k.\lambda}(\bar{\Omega})$ with $k \geq 0$ and $0 < \lambda \leq 1$. Then the unique (classical) solution of the boundary value problem (1.1) has the regularity $u \in C_{k+2.\lambda}$ and the a priori estimate holds true:

$$(1.3)\qquad \| u \|_{C_{k+2.\lambda}} \leq c \| f \|_{C_{k.\lambda}} \quad .$$

Now we assume that f is the divergence of a vector-valued function $\underline{F} = (F_1, F_2, \dots, F_n)$:

$$(1.4)\qquad f = -\nabla \cdot \underline{F} = -\sum_{(i)} F_{i|i} \quad .$$

The weak solution of (1.1) is characterized by

Definition 1.1: $u \in \mathring{H}_1$ is the weak solution of (1.1) if the variational equation

$$(1.5)\qquad \begin{aligned} D(u, \varphi) &= \sum_{(i)} (u_{|i}, \varphi_{|i}) \\ &= \sum_{(i)} (F_i, \varphi_{|i}) \end{aligned}$$

holds true for all test-functions $\varphi \in \mathcal{D}(\Omega)$.

Obviously the weak solution is well defined in case of $\underline{F} \in \underline{H}_0 = \underline{L}_2(\Omega)$, i. e. $F_i \in L_2(\Omega)$ for $i = 1, \dots, n$, and the following shift theorem is valid:

Theorem (S.3): Assume the right hand side $f = -\nabla \cdot \underline{F}$ has the regularity $\underline{F} \in \underline{H}_0$. Then the unique (weak) solution of the boundary value problem (1.1) has the regularity $u \in H_1$ and the a priori estimate holds true:

$$(1.6)\qquad \| u \|_{H_1} \leq c \| \underline{F} \|_{\underline{H}_0} = c \sum_{(i)} \| F_i \|_{L_2} \quad .$$

More general we will use

Definition 1.2: Let Ξ be a given (open) domain. A function $\tilde{u} \in H_1(\Xi)$ is a weak solution of the Poisson equation

$$(1.7) \qquad -\Delta u = -\nabla\cdot\underline{F} \quad \text{in } \Xi$$

if

$$(1.8) \qquad D(\tilde{u}, \varphi) = \sum_{(i)} (F_i, \varphi_{|i})$$

holds true for all test-functions $\varphi \in \mathcal{D}(\Xi)$.

The aim of this section is to give a direct proof of

Theorem 1.3: Assume the right hand side $f = -\nabla\cdot\underline{F}$ has the regularity $\underline{F} \in \underline{C}_{0,\lambda}$ (i. e. $F_i \in C_{0,\lambda}$ for $i = 1, \ldots, n$) with $0 < \lambda \leq 1$. Then the (weak) solution of the boundary value problem (1.1) has the regularity $u \in C_{1,\lambda}$ and the a priori estimate holds true:

$$(1.9) \qquad \| u \|_{C_{1,\lambda}} \leq c \| \underline{F} \|_{\underline{C}_{0,\lambda}} = c \sum_{(i)} \| F_i \|_{C_{0,\lambda}} .$$

Remark: In Appendix B we will show that the case of a right hand side of the structure

$$(1.4') \qquad f = -\nabla\cdot\underline{F} + F_0$$

with $F_0, F_1, \ldots, F_n \in C_{0,\lambda}$ is covered by Theorem 1.3, of course then (1.9) has to be changed to

$$(1.9') \qquad \| u \|_{C_{1,\lambda}} \leq c \{ \| F_0 \|_{C_{0,\lambda}} + \| \underline{F} \|_{\underline{C}_{0,\lambda}} \} .$$

As mentioned in the introduction the proof will follow the lines of Gilbarg-Trudinger. We begin with some lemmata.

Lemma 1.4: Let B_2, B_3 be concentric balls in $\mathbb{R}^n$ with center x^0 of radii $k_2 r$, $k_3 r$ respectively with $r > 0$ and $0 < k_2 < k_3$. Suppose $\underline{F} \in \underline{C}_{0,\lambda}(\bar{B}_3)$ with $0 < \lambda \leq 1$. There exists a weak solution $\tilde{u}$ of (1.7) in B_3, having the regularity $\tilde{u} \in C_{1,\lambda}(B_3)$ and admitting the a priori estimate ($c = c(r, k_2, k_3)$)

$$(1.10) \qquad \| \tilde{u} \|_{C_{1,\lambda}(\bar{B}_2)} \leq c \| \underline{F} \|_{\underline{C}_{0,\lambda}(\bar{B}_3)} .$$

Proof: Let $\Gamma(.)$ be the fundamental solution of the Laplace-operator (GT, p. 50). The functions

$$(1.11) \qquad \tilde{v}_j = \Gamma * F_j$$

i. e.

$$(1.12) \qquad \tilde{v}_j(x) = \int\int \Gamma(x-y) F_j(y)\, dy$$

(the domain of integration is B_3) are $C_{2,\lambda}(B_3)$-functions being solutions of

$$(1.13) \qquad -\Delta \tilde{v}_j = F_j \quad \text{in } B_3 ,$$

and the a priori estimates

$$(1.14) \qquad \| \tilde{v}_j \|_{C_{2,\lambda}(\bar{B}_2)} \leq c \| F_j \|_{C_{0,\lambda}(\bar{B}_3)}$$

are valid (GT, Lemma 4.4, p.56). Now we define

$$(1.15) \qquad \tilde{u}_j = -\tilde{v}_{j|j} .$$

Obviously we have $\tilde{u}_j \in C_{1,\lambda}(B_3)$ and

$$(1.16) \qquad \| \tilde{u}_j \|_{C_{1,\lambda}(\bar{B}_2)} \leq c \| F_j \|_{C_{0,\lambda}(\bar{B}_3)} .$$

With any $\varphi \in \mathcal{D}(B_3)$ we get

$$(1.17) \qquad \begin{aligned} D(\tilde{u}_j, \varphi) &= \sum_{(i)} (-\tilde{v}_{j|ji}, \varphi_{|i}) \\ &= \sum_{(i)} (-\tilde{v}_{j|ii}, \varphi_{|j}) \\ &= (-\Delta\tilde{v}_j, \varphi_{|j}) \\ &= (F_j, \varphi_{|j}) . \end{aligned}$$

This implies: The function

$$(1.18) \qquad \tilde{u} = \sum_{(j)} \tilde{u}_j$$

is a weak solution according to the assertions of Lemma 1.4. ∎

Lemma 1.5: Let B_1, B_3 be two concentric balls in $\mathbb{R}^n$ of radii $k_1 r$, $k_3 r$ respectively with $r > 0$, $0 < k_1 < k_3$. Suppose $\underline{F} \in \underline{C}_{0,\lambda}(\bar{B}_3)$ with $0 < \lambda \leq 1$. For any weak solution u of (1.7) the a priori estimate is valid with $c = c(r, k_1, k_3)$:

$$(1.19) \qquad \| u \|_{C_{1,\lambda}(\bar{B}_1)} \leq c \{ \| \underline{F} \|_{\underline{C}_{0,\lambda}(\bar{B}_3)} + \| u \|_{L_2(B_3)} \} .$$

Proof: Let $\tilde{u}$ be the function with the properties stated in Lemma 1.4 with e. g. $k_2 = (k_1 + k_3)/2$. The difference $v = u - \tilde{u}$ is harmonic in B_3 respectively in B_2. Therefore any (arbitrary strong) norm of v in B_1 is bounded by any (arbitrary weak) norm of v in B_2. ∎

In what follows $\mathbb{R}^n_+$ will denote the positive half-space, i.e. all $x \in \mathbb{R}^n$ with $x_n > 0$, and T will denote the hyperplane $x_n = 0$. For $x^0 \in T$ the half-balls $B_k \cap \mathbb{R}^n_+$ are denoted by B_k^+. Correspondingly T_k denotes the intersection $B_k \cap T$.

Lemma 1.6: Assume $x^0 \in T$ and $\underline{F} \in \underline{C}_{0,\lambda}(\bar{B}_3^+)$. There exists a function $\tilde{u} \in C_{1,\lambda}(B_3^+)$ with $\tilde{u} = 0$ on T_3 being a weak solution of (1.7) in B_3^+ such that the a priori estimate holds true:

$$(1.20) \qquad \| \tilde{u} \|_{C_{1,\lambda}(\bar{B}_2^+)} \leq c \| \underline{F} \|_{\underline{C}_{0,\lambda}(\bar{B}_3^+)} .$$

Proof: For any $y = (y_1, ..., y_n) \in \mathbb{R}^n_+$ we put $\bar{y} = (y_1, ..., y_{n-1}, -y_n)$ for the point reflected with respect to T. This time we define for $\alpha = 1, 2, ..., n-1$ the functions $\tilde{v}_\alpha$ by

$$(1.21) \qquad \tilde{v}_\alpha(x) = \int\int \Gamma(x-y) F_\alpha(y) dy - \int\int \Gamma(x-\bar{y}) F_\alpha(y) dy$$

and

$$(1.22) \qquad \tilde{v}_n(x) = \int\int \Gamma(x-y) F_n(y) dy + \int\int \Gamma(x-\bar{y}) F_n(y) dy$$

(the domain of integration is B_3^+). The function

$$(1.23) \qquad \tilde{u} = -\sum_{(j)} \tilde{v}_{j|j}$$

has the properties stated in Lemma 1.6. ∎

The proof of the next lemma follows the lines of the proof of Lemma 1.5, only Schwarz' reflection principle has to be applied in addition.

Lemma 1.7: Assume $x^0 \in T$ and $\underline{F} \in \underline{C}_{0,\lambda}(\bar{B}_3^+)$. For any weak solution u of (1.7) with $u \in \mathring{H}_1(B_3^+)$ the a priori estimate is valid:

$$(1.24) \qquad \| u \|_{C_{1,\lambda}(\bar{B}_1^+)} \leq c \{ \| \underline{F} \|_{\underline{C}_{0,\lambda}(\bar{B}_3^+)} + \| u \|_{L_2(B_3^+)} \} .$$

Lemma 1.5 and Lemma 1.7 are the counterparts of Theorem 4.6 (GT, p. 59) and Theorem 4.11 (GT, p. 63). The arguments of Gilbarg-Trudinger (GT, pp. 82-94) transferred to our setting finish the proof of Theorem 1.3 ∎

In the subsequent sections 2 and 3 we will need also a corresponding shift-theorem for inhomogeneous Dirichlet boundary conditions:

Theorem 1.8: Let the boundary value problem

$$(1.25) \qquad \begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega , \\ u &= U_D \quad \text{on } \partial\Omega \end{aligned}$$

be given. The regularity $U_D \in C_{1,\lambda}(\partial\Omega)$ implies the regularity $u \in C_{1,\lambda}(\bar{\Omega})$ of the solution, and the a priori estimate is valid:

$$(1.26) \qquad \| u \|_{C_{1,\lambda}(\bar{\Omega})} \leq c \| U_D \|_{C_{1,\lambda}(\partial\Omega)} .$$

Proof: In connection with the arguments of Gilbarg-Trudinger mentioned above it suffices to prove the counterparts of Lemmata 1.6, 1.7.

Lemma 1.9: Let $g \in C_{1,\lambda}(\bar{T}_3)$ be given. There exists a function $\tilde{u} \in C_{1,\lambda}(B_3^+)$ with

$$(1.27) \qquad \tilde{u} = g \quad \text{on } T_3 ,$$

harmonic in B_3^+, and admitting the a priori estimate

$$(1.28) \qquad \| \tilde{u} \|_{C_{1,\lambda}(\bar{B}_2^+)} \leq c \| g \|_{C_{1,\lambda}(\bar{T}_3)} .$$

Proof: We define the function $G \in C_{1,\lambda}(B_3)$ by

(1.29) $$G(x_1,\dots,x_{n-1},x_n) = g(x_1,\dots,x_{n-1})$$

and consider the difference

(1.30) $$\tilde{w} = u - G .$$

In order that $\tilde{u}$ has the properties stated in the lemma it is sufficient to show the existence of a function $\tilde{w}$ according to

(1.31) $$\begin{aligned} -\Delta\tilde{w} &= \Delta G \quad \text{in} \quad B_3^\circ , \\ \tilde{w} &= 0 \quad \text{on} \quad T_3 , \end{aligned}$$

such that the estimate

(1.32) $$\| \tilde{w} \|_{C_{1,\lambda}(\bar{B}_2^\circ)} \le c \| g \|_{C_{1,\lambda}(\bar{T}_3)} .$$

holds true. We define the function $\underline{F}$ by

(1.33) $$\begin{aligned} F_\alpha &= G_{|\alpha} = g_{|\alpha} \quad \text{for} \quad \alpha = 1,\dots, n-1 , \\ F_n &= 0 . \end{aligned}$$

Obviously we have $\underline{F} \in \underline{C}_{0,\lambda}(\bar{B}_3^\circ)$ and

(1.34) $$\| \underline{F} \|_{\underline{C}_{0,\lambda}(\bar{B}_3^\circ)} \le c \| g \|_{C_{1,\lambda}(\bar{T}_3)} .$$

Thus Lemma 1.6 guarantees the existence of a weak solution $\tilde{w}$ according to the regularity and the a priori estimates stated. ∎

Lemma 1.10: Let $g \in C_{1,\lambda}(\bar{T}_3)$ be given and let u be a weak solution of

(1.35) $$-\Delta u = 0 \quad \text{in} \quad B_3^\circ$$

with boundary values

(1.36) $$u = g \quad \text{on} \quad T_3 .$$

Then the a priori estimate is valid:

(1.37) $$\| u \|_{C_{1,\lambda}(\bar{B}_1^\circ)} \le c \{ \| g \|_{C_{1,\lambda}(\bar{B}_3)} + \| u \|_{L_2(B_3^\circ)} \} .$$

The proof follows the arguments given above in connection with Lemma 1.7. This finishes the proof of Theorem 1.8. ∎

2. In this section we consider the problem

(2.1) $$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega , \\ u &= 0 \\ u_{|n} &= 0 \end{aligned} \Bigg\} \text{ on } \partial\Omega .$$

We will not repeat the counterparts of the Shift-Theorems (S.1) and (S.2) and the corresponding norm estimates. In the present case the "2" in (1.2) and (1.3) is to be replaced by "4".

This time we consider right hand sides f (2.1) of the structure

(2.2) $$f = \nabla^2 \cdot \underline{F} = \sum_{(i,j)} F_{ij|ij} .$$

Then the weak solution of (2.1) is characterized by

Definition 2.1: $u \in \mathring{H}_2$ is the weak solution of (2.1) if the variational equation

(2.3) $$\begin{aligned} (\Delta u , \Delta\varphi) &= \sum_{(i,j)} (u_{|ij} , \varphi_{|ij}) \\ &= \sum_{(i,j)} (F_{ij} , \varphi_{|ij}) \end{aligned}$$

holds true for all test-functions $\varphi \in \mathcal{D}(\Omega)$.

Obviously the weak solution is well defined in case of $\underline{F} \in \underline{H}_0 = \underline{L}_2(\Omega)$, i. e. $F_{ij} \in L_2(\Omega)$ for $i,j = 1,\dots,n$, and the following shift-theorem is valid:

Theorem (S.4): Assume the right hand side $f = \nabla^2 \cdot \underline{F}$ in (2.1) has the regularity $\underline{F} \in \underline{H}_0$. Then the (unique) weak solution of the boundary value problem (2.1) has the regularity $u \in H_2$ and the a priori estimate holds true:

$$(2.4) \qquad \| u \|_{H_2} \le c \| \underline{F} \|_{\underline{H}_0} = c \sum_{(i,j)} \| F_{ij} \|_{L_2} .$$

The counterpart of Theorem (1.3) is

Theorem 2.2: Assume the right hand side $f = \nabla^2 \cdot \underline{F}$ has the regularity $\underline{F} \in \underline{C}_{0.\lambda}$, i. e. $F_{ij} \in C_{0.\lambda}$ for $i, j = 1,\dots,n$. Then the weak solution of the boundary value problem (2.1) has the regularity $u \in C_{2.\lambda}$ and the a priori estimate holds true:

$$(2.5) \qquad \| u \|_{C_{2.\lambda}} \le c \| \underline{F} \|_{\underline{C}_{0.\lambda}} = c \sum_{(i,j)} \| F_{ij} \|_{C_{0.\lambda}} .$$

Remark: Analogue to the situation for second order equations – see the remark following Theorem 1.3 – we could consider right hand sides f (2.2) of the structure

$$(2.2') \qquad f = \nabla^2 \cdot \underline{F} - \nabla \cdot \underline{F}_0 + F_{00}$$

with $F_{\alpha\beta} \in C_{0.\lambda}$ for $\alpha, \beta = 0, 1, \dots, n$. Then (2.5) has to be changed to

$$(2.5') \qquad \| u \|_{C_{2.\lambda}} \le c \{ \| \underline{F} \|_{\underline{C}_{0.\lambda}} + \| \underline{F}_0 \|_{\underline{C}_{0.\lambda}} + \| F_{00} \|_{C_{0.\lambda}} \} .$$

The proof of Theorem 2.2 could follow the lines of section 1 but we will use a different approach. First we show

Lemma 2.3: Let B_2, B_3 and λ be as stated in Lemma 1.4 and let $\underline{F} \in \underline{C}_{0.\lambda}(\bar{B}_3)$ be given. Then there exists a weak solution $\tilde{u}$ of (2.1_1) in B_3, having the regularity $\tilde{u} \in C_{2.\lambda}(B_3)$ and admitting the estimate

$$(2.6) \qquad \| \tilde{u} \|_{C_{2.\lambda}(\bar{B}_2)} \le c \| \underline{F} \|_{\underline{C}_{0.\lambda}(\bar{B}_3)} .$$

Proof: Similar to the proof of Lemma 1.4 we introduce the functions

$$(2.7) \qquad \begin{aligned} \tilde{v}_{ij} &= \Gamma * F_{ij} , \\ \tilde{w}_{ij} &= \tilde{v}_{ij|j} , \\ \tilde{z}_{ij} &= \Gamma * \tilde{w}_{ij} , \\ \tilde{u}_{ij} &= \tilde{z}_{ij|i} . \end{aligned}$$

With $\varphi \in \mathcal{D}(B_3)$ we get by partial integration the sequence

$$(2.8) \qquad \begin{aligned} (\Delta\tilde{u}_{ij}, \Delta\varphi) &= (\tilde{u}_{ij}, \Delta^2\varphi) \\ &= -(\tilde{z}_{ij}, \Delta^2\varphi_{|i}) = -(\Delta\tilde{z}_{ij}, \Delta\varphi_{|i}) \\ &= (\tilde{w}_{ij}, \Delta\varphi_{|i}) = -(\tilde{v}_{ij}, \Delta\varphi_{|ij}) \\ &= -(\Delta\tilde{v}_{ij}, \varphi_{|ij}) = (F_{ij}, \varphi_{|ij}) . \end{aligned}$$

Now we define

$$(2.9) \qquad \tilde{u} = \sum_{(i,j)} \tilde{u}_{ij} .$$

Obviously $\tilde{u}$ is a weak solution of (2.1_1) in B_3. We conclude $\tilde{u} \in C_{2.\lambda}(B_3)$ and

$$(2.10) \qquad \| \tilde{u} \|_{C_{2.\lambda}(\bar{B}_2)} \le c \| \underline{F} \|_{\underline{C}_{0.\lambda}(\bar{B}_3)} . \quad ■$$

Now we turn over to a weakened version of Theorem 2.2:

Lemma 2.4: Under the assumptions of Theorem 2.2 there exists a weak solution $\tilde{u}$ of the partial differential equation (2.1_1) such that (2.5) holds true with u replaced by $\tilde{u}$.

Proof: Let ${}^e\Omega$ be chosen fixed such that the inclusion $\Omega \subset\subset {}^e\Omega$ holds; e. g. we may use (for some $h > 0$)

$$(2.11) \qquad {}^e\Omega = \{ x \mid \operatorname{dist}(x, \bar{\Omega}) < h \} .$$

Any function $g \in C_{k.\lambda}(\bar{\Omega})$ can be extended to a function ${}^e g \in C_{k.\lambda}({}^e\bar{\Omega})$ such that

$$(2.12) \qquad \| {}^e g \|_{C_{k.\lambda}({}^e\bar{\Omega})} \le c \| g \|_{C_{k.\lambda}(\bar{\Omega})}$$

– see Stein (1970), p. 175, p. 194 –. Now let ${}^e\underline{F}$ be a corresponding extension of $\underline{F}$ to ${}^e\Omega$. In view of Lemma 2.3 in combination with the arguments of Gilbarg-Trudinger mentioned above there exists a weak solution ${}^e\tilde{u}$ of

(2.13) $$\Delta^2 u = \nabla^2 \cdot \underline{F} \quad \text{in } {}^e\Omega$$

with ${}^e\tilde{u} \in C_{2,\lambda}({}^e\Omega)$ and

(2.14) $$\begin{aligned} \| {}^e\tilde{u} \|_{C_{2,\lambda}(\bar\Omega)} &\le c \| {}^e\underline{F} \|_{\underline{C}_{0,\lambda}({}^e\bar\Omega)} \\ &\le c \| \underline{F} \|_{\underline{C}_{0,\lambda}(\bar\Omega)} \quad . \end{aligned}$$

Thus the function $\tilde{u} = {}^e\tilde{u}_{|\Omega}$ has the properties stated in Lemma 2.4. ∎

Now we turn over to the proof of Theorem 2.2. Let $\tilde{u}$ be a function guaranteed by Theorem 2.4. With u being the solution of the original boundary value problem (2.1) we put

(2.15) $$U = \tilde{u} - u \quad .$$

Then U has to be a solution of

(2.16) $$\begin{aligned} \Delta^2 U &= 0 && \text{in } \Omega \quad , \\ U &= U_D && \\ U_n &= U_N && \text{on } \partial\Omega \quad . \end{aligned}$$

Here U_D respective U_N denote the trace of $\tilde{u}$ respective of its normal derivative on $\partial\Omega$. We have the regularity $U_D \in C_{2,\lambda}(\partial\Omega)$ respective $U_N \in C_{1,\lambda}(\partial\Omega)$ and the a priori bound

(2.17) $$\| U_D \|_{C_{2,\lambda}(\partial\Omega)} + \| U_N \|_{C_{1,\lambda}(\partial\Omega)} \le c \| \underline{F} \|_{\underline{C}_{0,\lambda}} \quad .$$

In order to finish the proof of Theorem 2.2 it remains to show

Theorem 2.5: Under the assumptions $U_D \in C_{2,\lambda}(\partial\Omega)$, $U_N \in C_{1,\lambda}(\partial\Omega)$ the solution of the boundary value problem (2.16) has the regularity $U \in C_\infty(\Omega) \cap C_{2,\lambda}(\bar\Omega)$ and admits the a priori estimate

(2.18) $$\| U \|_{C_{2,\lambda}} \le c \{ \| U_D \|_{C_{2,\lambda}(\partial\Omega)} + \| U_N \|_{C_{1,\lambda}(\partial\Omega)} \} \quad .$$

Proof: The fact $U \in C_\infty(\Omega)$ is a consequence of Weyl's lemma and is not discussed here. In connection with standard arguments as already used above it is sufficient to prove a local version of Theorem 2.5. We will adopt the notations of section 1, this time we will work with four different concentric balls with center $x^0 \in T$.

Lemma 2.6: Let $g \in C_{2,\lambda}(\bar T_4)$, $h \in C_{1,\lambda}(\bar T_4)$ be given. There exists a function $\tilde{u} \in C_{2,\lambda}(B_4^+)$, biharmonic in B_4^+ with the boundary values

(2.19) $$\begin{aligned} \tilde{u} &= g \\ \tilde{u}_{|n} &= h \end{aligned} \quad \text{on } T_4$$

and admitting the a priori estimate

(2.20) $$\| \tilde{u} \|_{C_{2,\lambda}(\bar B_1^+)} \le c \{ \|g\|_{C_{2,\lambda}(\bar T_4)} + \|h\|_{C_{1,\lambda}(\bar T_4)} + \|\tilde{u}\|_{L_2(B_4^+)} \} \,.$$

Proof: We will use the classical result:

Theorem (R): Let u be biharmonic in the halfball B^+. There exist two functions v, w harmonic in B^+ such that the representation is valid:

(2.21) $$u = v + x_n w \quad .$$

For the sake of completeness we reproduce the proof to be found e. g. in Frank – v. Mises (1930), p. 848 in Appendix A.

Our aim is to construct functions $\tilde{v}$, $\tilde{w}$ such that by (2.21) a function $\tilde{u}$ is given according to Lemma 2.6. In the first step we consider the problem

(2.22) $$\begin{aligned} -\Delta v &= 0 && \text{in } B_4^+ \quad , \\ v &= g && \text{on } T_4 \quad . \end{aligned}$$

Theorem 6.6 (GT, p. 93) guarantees the existence of a solution $\tilde{v}$ of (2.22) with $\tilde{v} \in C_{2,\lambda}(B_4^+)$ and

(2.23) $$\| \tilde{v} \|_{C_{2,\lambda}(\bar B_3^+)} \le c \| g \|_{C_{2,\lambda}(\bar T_4)} \quad .$$

In the second step we consider the problem

$$(2.24)\qquad \begin{aligned} -\Delta w &= 0 && \text{in } B_3^{\circ} \quad ,\\ w &= \hat{h} = h - \tilde{v}_{|n} && \text{on } T_3 \quad . \end{aligned}$$

In view of (2.23) and our assumptions on g, h we have

$$(2.25)\qquad \| \hat{h} \|_{C_{1,\lambda}(\bar{T}_3)} \le c\{\| g \|_{C_{2,\lambda}(\bar{T}_4)} + \| h \|_{C_{1,\lambda}(\bar{T}_4)}\} .$$

Lemma 1.9 guarantees the extistence of a solution $\tilde{w} \in C_{1,\lambda}(B_3^{\circ})$ with

$$(2.26)\qquad \| \tilde{w} \|_{C_{1,\lambda}(\bar{B}_2^{\circ})} \le c\{\| g \|_{C_{2,\lambda}(\bar{T}_4)} + \| h \|_{C_{1,\lambda}(\bar{T}_4)}\} .$$

Since $\tilde{v}$, $\tilde{w}$ are harmonic the function

$$(2.27)\qquad \tilde{u} = \tilde{v} + x_n \tilde{w}$$

is biharmonic in B_2°. Moreover $\tilde{u}$ has the properties: (i) The boundary conditions ($2.16_{2,3}$) are fulfilled; (ii) In view of (2.23) and (2.26) the function $\tilde{u}$ has the regularity $\tilde{u} \in C_{1,\lambda}(B_3^{\circ})$. - Actually $\tilde{u}$ possesses a higher regularity than stated in (ii). In order to see this we look at the differential equation

$$(2.28)\qquad -\Delta\tilde{u} = -2\tilde{w}_{|n}$$

which is a consequence of the representation (2.27) with $\tilde{v}$, $\tilde{w}$ being harmonic functions. This implies: $\tilde{u}$ (2.27) is a solution of the boundary value problem

$$(2.29)\qquad \begin{aligned} -\Delta u &= f = -2\tilde{w}_{|n} && \text{in } B_2^{\circ} \quad ,\\ u &= g && \text{on } T_2 \quad . \end{aligned}$$

We have the regularity $f \in C_{0,\lambda}(\bar{B}_2^{\circ})$ and $g \in C_{2,\lambda}(\bar{T}_2)$. Standard a priori estimates {GT, p.66} then lead to $\tilde{u} \in C_{2,\lambda}(\bar{B}_2^{\circ})$ and to the bound (2.20). ∎

3. Now we turn over to the discussion of regularity results similar to above for Stokes' flows. We will restrict ourselves to n = 2 dimensions. Slightly generalizing we consider the following boundary value problem:

$$(3.1)\qquad \left.\begin{aligned} -\Delta\underline{u} + \nabla p &= \underline{f} \\ \nabla\cdot\underline{u} &= h \end{aligned}\right\} \text{ in } \Omega \ ,\qquad \underline{u} = 0 \quad \text{on } \partial\Omega \ .$$

Obviously the function p (physically the pressure) is arbitrary with respect to an additive constant. As customary $\dot{L}_2$ denotes the factor space $L_2/\mathbb{R}$ equipped with the corresponding factor norm.

Corresponding to above we consider right hand sides $\underline{f}$ (3.1_1) with a reduced regularity. This time we assume

$$(3.2)\qquad \underline{f} = -\nabla\cdot\underline{\sigma} \quad ,$$

i.e.

$$(3.2')\qquad f_i = -\sum_{(j)} \sigma_{ij|j} \quad .$$

The range of the indices i,j is {1, 2} . For right hand sides $\underline{f}$ of the structure (3.2) the weak solution of (3.1) is characterized by

Definition 3.1: $\underline{u} \in \mathring{H}_1$ is the weak solution of (3.1) if the variational equations

$$(3.3)\qquad \begin{aligned} (\nabla\underline{u}, \nabla\underline{v}) - (p, \nabla\cdot\underline{v}) &= (\underline{\sigma}, \nabla\underline{v}) \quad ,\\ (q, \nabla\cdot\underline{u}) &= (q, h) \end{aligned}$$

hold true for all test-functions $\underline{v} \in \mathcal{D}(\Omega)$ and $q \in \dot{L}_2 = \dot{L}_2(\Omega)$.

The inner products in (3.3) are defined by

$$(3.4)\qquad \begin{aligned} (\nabla\underline{u}, \nabla\underline{v}) &= \sum_{(i,j)} (u_{i|j}, v_{i|j}) \quad ,\\ (p, \nabla\cdot\underline{v}) &= \sum_{(j)} (p, v_{j|j}) \quad ,\\ (\underline{\sigma}, \nabla\underline{v}) &= \sum_{(i,j)} (\sigma_{ij}, v_{i|j}) \quad . \end{aligned}$$

In the situation at present we have the shift-theorem corresponding to S.3 and S.4

Theorem (S.5): Assume the regularity $\underline{f} = -\nabla\cdot\underline{\underline{\sigma}}$ with $\underline{\underline{\sigma}} \in \underline{\underline{H}}_0$ and $h \in \mathring{L}_2$. Then the unique (weak) solution $\{\underline{u}, p\}$ of the boundary value problem (3.1) has the regularity $\underline{u} \in \underline{H}_1$, $p \in \mathring{L}_2$ and the a priori estimate holds true:

$$(3.5) \qquad \|\underline{u}\|_{\underline{H}_1} + \|p\|_{\mathring{L}_2} \quad \le \quad c\,\{\|\underline{\underline{\sigma}}\|_{\underline{\underline{H}}_0} + \|h\|_{\mathring{L}_2}\} \quad .$$

In correspondence with Theorem 1.3 and Theorem 2.2 we will give a direct proof of

Theorem 3.2: Assume $\underline{f} = -\nabla\cdot\underline{\underline{\sigma}}$ with $\underline{\underline{\sigma}} \in \underline{\underline{C}}_{0,\lambda}$ and $h \in \mathring{C}_{0,\lambda} = C_{0,\lambda} \cap \mathring{L}_2$. Then the unique (weak) solution $\{\underline{u}, p\}$ of the boundary value problem (3.1) has the regularity $\underline{u} \in \underline{C}_{1,\lambda}$, $p \in \mathring{C}_{0,\lambda}$, and the a priori estimate holds true:

$$(3.6) \qquad \|\underline{u}\|_{\underline{C}_{1,\lambda}} + \|p\|_{\mathring{C}_{0,\lambda}} \le \quad c\,\{\|\underline{\underline{\sigma}}\|_{\underline{\underline{C}}_{0,\lambda}} + \|h\|_{\mathring{C}_{0,\lambda}}\} \quad .$$

First we will prove a weakened version (see Lemma 2.4):

Lemma 3.3: Under the assumptions of Theorem 3.2 there exists a weak solution $\{\tilde{\underline{u}}, \tilde{p}\}$ of the partial differential equations $(3.1_{1,2})$ such that (3.6) holds true with $\{\underline{u}, p\}$ replaced by $\{\tilde{\underline{u}}, \tilde{p}\}$.

Proof: Let ${}^e\Omega$ be an extension of Ω (see (2.11)) and ${}^e\underline{\underline{\sigma}}$, ${}^e h$ be corresponding extensions of $\underline{\underline{\sigma}}$, h to ${}^e\Omega$ such that estimates of the type (2.12) are valid. In order to show the existence of $\{\tilde{\underline{u}}, \tilde{p}\}$ we use the "Ansatz"

$$(3.7) \qquad \begin{aligned} \tilde{u}_1 &= \psi_{|x} - \varphi_{|y} \quad , \\ \tilde{u}_2 &= \psi_{|y} + \varphi_{|x} \end{aligned}$$

with two functions ψ, φ. Here x, y are the two independent variables x_1, x_2. In addition we introduce

$$(3.8) \qquad \vartheta = -\Delta\varphi \quad .$$

In terms of φ, ψ and ϑ the differential equations $(3.1_{1,2})$ are

$$(3.9) \qquad \begin{aligned} -\Delta\psi_{|x} - \vartheta_{|y} + p_{|x} &= {}^e\sigma_{11|x} + {}^e\sigma_{12|y} \quad , \\ -\Delta\psi_{|y} + \vartheta_{|x} + p_{|y} &= {}^e\sigma_{21|x} + {}^e\sigma_{22|y} \quad , \\ \Delta\psi &= {}^e h \quad . \end{aligned}$$

These 3 equations can be decoupled into

$$(3.10) \qquad -\Delta\psi = -{}^e h \quad ,$$

$$(3.11) \qquad \begin{aligned} -\vartheta_{|y} + p_{|x} &= ({}^e\sigma_{11} + {}^e h)_{|x} + {}^e\sigma_{12|y} \quad , \\ \vartheta_{|x} + p_{|y} &= {}^e\sigma_{21|x} + ({}^e\sigma_{22} + {}^e h)_{|y} \quad . \end{aligned}$$

By Theorem S.2 the existence of a solution ψ of (3.10) with the regularity $\psi \in C_{2,\lambda}({}^e\Omega)$ and

$$(3.12) \qquad \|\psi\|_{C_{2,\lambda}(\bar{\Omega})} \quad \le \quad c\,\|{}^e h\|_{C_{0,\lambda}({}^e\bar{\Omega})} \quad \le \quad c\,\|h\|_{\mathring{C}_{0,\lambda}(\bar{\Omega})}$$

is guaranteed. A weak solution of the Cauchy-Riemann-equations (3.11) is given by

$$(3.13) \qquad \begin{aligned} \vartheta &= A_{11|xy} + A_{12|yy} - A_{21|xx} - A_{22|xy} \quad , \\ p &= -A_{11|xx} - A_{12|xy} - A_{21|xy} - A_{22|yy} \end{aligned}$$

with

$$(3.14) \qquad A_{ij} = \Gamma * ({}^e\sigma_{ij} + {}^e h\,\delta_{ij}) \quad .$$

The convolution $\Gamma * (\cdot)$ has to be taken over the domain ${}^e\Omega$. Further let φ be a weak solution of (3.8) in ${}^e\Omega$. In view of the general a priori estimates we conclude $\varphi \in C_{2,\lambda}({}^e\Omega)$, $p \in C_{0,\lambda}({}^e\Omega)$ and

$$(3.15) \qquad \|\varphi_{|\Omega}\|_{C_{2,\lambda}} + \|p_{|\Omega}\|_{C_{0,\lambda}} \le \quad c\,\{\|\underline{\underline{\sigma}}\|_{\underline{\underline{C}}_{0,\lambda}} + \|h\|_{\mathring{C}_{0,\lambda}}\} \quad .$$

By this and (3.12) the proof of Lemma 3.3 is finished. ■

Now we consider the difference

$$(3.16) \qquad \underline{U} = \underline{u} - \tilde{\underline{u}} \quad , \quad P = p - \tilde{p}$$

between the solution of the boundary value problem (3.1) and $\{\underline{\tilde u}, \tilde p\}$ just constructed. $\{\underline{U}, P\}$ has to be a solution of

$$(3.17)\quad \left.\begin{aligned} -\Delta \underline{U} + \nabla P &= 0 \\ \nabla\cdot\underline{U} &= 0 \end{aligned}\right\} \text{ in } \Omega, \qquad \underline{U} = -\underline{\tilde u} \quad \text{on } \partial\Omega .$$

Similar to (3.7) we use the "Ansatz"

$$(3.18)\qquad U_1 = -\Phi_{|y} \quad , \quad U_2 = \Phi_{|x} .$$

In this way the condition of incompressibility (3.17_2) is fulfilled. The equations (3.17_1) are in terms of Φ and P

$$(3.19)\qquad \begin{aligned} (\Delta\Phi)_{|y} + P_{|x} &= 0 \quad , \\ -(\Delta\Phi)_{|x} + P_{|y} &= 0 \quad . \end{aligned}$$

Necessarily Φ has to be biharmonic in Ω:

$$(3.20)\qquad \Delta^2\Phi = 0 \quad \text{in } \Omega .$$

Then P is defined by (3.19) up to a constant, i.e. $P \in \mathring{L}_2$ is uniquely defined. Now we look at the boundary conditions for Φ. Because of (3.7) these are

$$(3.21)\qquad \left.\begin{aligned} \Phi_{|x} &= -\varphi_{|x} - \psi_{|y} \\ \Phi_{|y} &= -\varphi_{|y} + \psi_{|x} \end{aligned}\right\} \text{on } \partial\Omega .$$

In terms of the tangential derivative "$\cdot_{|s}$" (s denotes the arc length) and the normal derivative "$\cdot_{|n}$" the conditions (3.21) are

$$(3.22)\qquad \begin{aligned} \Phi_{|s} &= -\varphi_{|s} - \psi_{|n} \quad , \\ \Phi_{|n} &= -\varphi_{|n} + \psi_{|s} \quad . \end{aligned}$$

The right hand sides in (3.22) are $C_{1,\lambda}(\partial\Omega)$ functions because of (3.12) and (3.15). In addition we have the bounds

$$(3.23)\quad \|\varphi_{|s} + \psi_{|n}\|_{C_{1,\lambda}(\partial\Omega)} + \|\varphi_{|n} - \psi_{|s}\|_{C_{1,\lambda}(\partial\Omega)} \le c\{\|\underline{g}\|_{\underline{C}_{0,\lambda}} + \|h\|_{\mathring{C}_{0,\lambda}}\} .$$

(3.22_1) can be rewritten as a condition on Φ because of the following fact: The function ψ is defined by (3.10). Because of the assumption $h \in \mathring{L}_2$ we have

$$(3.24)\qquad \oint_{\partial\Omega} \psi_{|n} = -\iint_{\Omega} h = 0 .$$

Therefore there exists a function χ, unique up to a constant, such that

$$(3.25)\qquad \chi_{|s} = \psi_{|n} .$$

We may normalize χ such that

$$(3.26)\qquad \oint_{\partial\Omega} \chi = 0 .$$

The regularity of ψ leads to $\chi \in C_{2,\lambda}(\partial\Omega)$. Therefore the boundary conditions (3.21) have the structure

$$(3.27)\qquad \left.\begin{aligned} \Phi &= \Phi_D = -\varphi - \chi \\ \Phi_{|n} &= \Phi_N = -\varphi_{|n} + \psi_{|s} \end{aligned}\right\} \text{on } \partial\Omega$$

with the regularity $\Phi_D \in C_{2,\lambda}(\partial\Omega)$, $\Phi_N \in C_{1,\lambda}(\partial\Omega)$ and the bound

$$(3.28)\quad \|\Phi_D\|_{C_{2,\lambda}(\partial\Omega)} + \|\Phi_N\|_{C_{1,\lambda}(\partial\Omega)} \le c\{\|\underline{g}\|_{\underline{C}_{0,\lambda}} + \|h\|_{\mathring{C}_{0,\lambda}}\} .$$

By the aid of Theorem 2.5 we conclude: The solution Φ of the boundary value problem (3.20), (3.21), (3.22) respective (3.21), (3.27) has the regularity $\Phi \in C_{2,\lambda}$, the norm is bounded by – choosing the additive constant appropriately

$$(3.29)\qquad \|\Phi\|_{C_{2,\lambda}} \le c\{\|\underline{g}\|_{\underline{C}_{0,\lambda}} + \|h\|_{\mathring{C}_{0,\lambda}}\} .$$

Going back to the definition of $\underline{U}$ (3.17), (3.18) and taking into account the estimates of $\underline{\tilde u}$ – see Lemma 3.3 – the bound (3.6) concerning $\underline{u}$ is proved.

It remains to consider the function P defined by (3.19). For any domain $\Omega' \subset\subset \Omega$ the bound

$$(3.30)\qquad \|P\|_{C_{0,\lambda}(\Omega')} \le c\|\Phi\|_{C_{2,\lambda}}$$

with $c = c(\Omega')$ is obvious. Thus it is only necessary to consider domains "near" to $\partial\Omega$. In view of the typical arguments mentioned above it suffices to prove the following lemma which may be considered as a counterpart of

Lemma 1.6. Here B_i denote balls of radii $r_i > 0$ with center $x^0 \in T$, B_i^* the upper half-balls and T_i the corresponding intersections with T.

<u>Lemma 3.4:</u> Let Φ be given biharmonic in B_3^* and with the regularity $\Phi \in C_{2,\lambda}(\bar{B}_3^*)$. There exists a function P being a solution of the Cauchy-Riemann-equations (3.19) which admits choosing the additive constant appropriately the estimate:

(3.31) $$\| P \|_{C_{0,\lambda}(\bar{B}_1^*)} \leq c \| \Phi \|_{C_{2,\lambda}(\bar{B}_3^*)} .$$

<u>Proof:</u> We put $Q = \Delta\Phi$. In terms of the theory of one complex variable in order to solve (3.19) we have to find the imaginary part P of an analytic function F(z) (with $z = x + iy$)

(3.32) $$F(z) = Q(z) + i\,P(z)$$

such that F is analytic in B_3^*. We consider the function

(3.33) $$\tilde{F}(z) = \pi^{-1} \int (\xi - z)^{-1} Q(\xi,0)\, d\xi =: \tilde{Q}(z) + i\,\tilde{P}(z) .$$

(The interval of integration is T_3). Obviously $\tilde{F}$ is analytic in B_3^*, and the real part $\tilde{Q}$ coincides with Q on T_3. In the standard way the estimates

(3.34) $$\| \tilde{Q} \|_{C_{0,\lambda}(\bar{B}_2^*)} + \| \tilde{P} \|_{C_{0,\lambda}(\bar{B}_2^*)} \leq c \| Q \|_{C_{0,\lambda}(\bar{T}_3)} \leq c \| \Phi \|_{C_{2,\lambda}(\bar{B}_{3,\lambda})}$$

are derived. The difference $q := \tilde{Q} - Q$ is harmonic in B_3^* and vanishes on T_3. By Schwarz' reflection principle q can be extended to a harmonic (continuous) function in all of B_3. Because of Weyl's lemma any (arbitrary strong) norm of q in B_2 is bounded by any (arbitrary weak) norm of q in B_3. Using the Cauchy- resp. Poisson-integral-formula for the domain B_2 we conclude: There exists an analytic function

(3.35) $$f(z) = q(z) + i\,p(z)$$

defined in B_2 such that the $C_{0,\lambda}(\bar{B}_1^*)$-norm of p is bounded according to (3.31). The function $P = \tilde{P} + p$ has the properties stated in the lemma. ∎

<u>**Appendix A:**</u> (see Frank - v. Mises (1930))
Let u be biharmonic in the half ball B^*. Our aim is to construct two functions v,w harmonic in B^* such that the representation

(A.1) $$u = v + x_n\, w$$

is valid. We will use for $x \in \mathbb{R}^n$ the splitting $x = (\dot{x}, \xi)$ with $\dot{x} \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}$. Partial differentiation of a function $z = z(\dot{x}, \xi)$ with respect to $\xi = x_n$ is denoted by $z_{|n} = \partial_\xi z$, if necessary we write also $(\partial_\xi z)(\dot{x}, \xi)$ etc.. The Laplace-operator with respect to the first (n-1) variables is denoted by

(A.2) $$\Delta^* z = \sum_{\alpha=1}^{n-1} z_{|\alpha\alpha} .$$

If v and w are harmonic, as we assume for the moment, we get from (A.1)

(A.3) $$\Delta u = 2w_{|n} = 2\partial_\xi w .$$

Thus we have the necessary condition for w:

(A.4) $$2\partial_\xi w = \Delta u .$$

For convenience we introduce the abbreviation $P = \Delta u$. From (A.4) it follows

(A.5) $$2w(\dot{x}, \xi) = \int_0^\xi P(\dot{x}, \eta)\, d\eta + W(\dot{x})$$

with W depending only on the n-1 variables $\dot{x}$. By applying the Laplace operator we get from (A.5)

(A.6) $$2(\Delta w)(\dot{x}, \xi) = \int_0^\xi (\Delta^* P)(\dot{x}, \eta)\, d\eta + (\partial_\xi P)(\dot{x}, \xi) + (\Delta^* W)(\dot{x}) .$$

Since $P = \Delta u$ is harmonic it is

(A.7) $$\Delta^* P = -\partial_\xi^2 P .$$

With (A.7) we find

(A.8) $\int_0^{\xi} (\Delta^* P)(\dot{x}, \eta)\, d\eta \;=\; -(\partial_\xi P)(\dot{x}, \xi) + (\partial_\xi P)(\dot{x}, 0)$

and further

(A.9) $2(\Delta w)(\dot{x}, \xi) \;=\; (\partial_\xi P)(\dot{x}, 0) + (\Delta^* W)(\dot{x})$.

Now let $W = W(\dot{x})$ be a particular solution of

(A.10) $\Delta^* W \;=\; -(\partial_\xi P)(\dot{x}, 0)$

which always exists. Then the function w defined by (A.5) is harmonic. It remains to show that the difference

(A.11) $v \;=\; u - x_n W$.

is harmonic. Because of $\Delta w = 0$ we get

(A.12)
$$\Delta v \;=\; \Delta u - 2 w_{1n} \;=\; P - 2\partial_\xi W \; .$$

By the very construction of w – see (A.4), (A.5) – the right hand side in (A.12) vanishes. ∎

In the above derivation we did not take care of any regularity assumptions. We could have thought of functions sufficiently smooth. Actually in section 2 the representation of a biharmonic function was just the motivation for introducing v, w resp. $\tilde{v}$, $\tilde{w}$ etc..

Appendix B:

Let $F_0 \in C_{0,\lambda}$ be given. It is possible to choose a function $\underline{G} \in \underline{C}_{0,\lambda}$ such that F_0 admits the representation

(B.1) $F_0 \;=\; -\nabla \cdot \underline{G}$

and

(B.2) $\| \underline{G} \|_{C_{0,\lambda}} \;\le\; c \| F_0 \|_{C_{0,\lambda}}$

holds true.

Without loss of generality we may assume that Ω is contained in the unit cube

(B.3) $Q \;=\; \{ x \mid 0 < x_i < 1 \text{ for } i = 1,2, \dots, n \}$.

In order to construct $\underline{G}$ we extend F_0 to ${}^e F_0$ defined in Q – see Stein (1970), p. 175, p. 194 – such that

(B.4) $\| {}^e F_0 \|_{C_{0,\lambda}(\bar{Q})} \;\le\; c \| F_0 \|_{C_{0,\lambda}}$

holds true. Using the splitting $x = (\dot{x}, \xi)$ of Appendix A we define $\underline{G}$ by

(B.5)
$$G_i(x) \;=\; \begin{cases} 0 & \text{for } i = 1, \dots, n-1 , \\ -\int_0^{x_n} F_0(\dot{x}, \xi)\, d\xi & \text{for } i = n \end{cases} \; .$$

Obviously $\underline{G}$ restricted to Ω has the properties (B.1), (B.2).

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Prof. Dr. Joachim A. Nitsche
Institut für Angewandte Mathematik
Hermann-Herder-Straße 10
7800 - Freiburg i. Breisgau
Bundesrepublik Deutschland