CHAPTER 3

Harmonic Analysis Tools for Solving the Incompressible Navier–Stokes Equations

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Introduction

Formulated and intensively studied at the beginning of the nineteenth century, the classical partial differential equations of mathematical physics represent the foundation of our knowledge of waves, heat conduction, hydrodynamics and other physical problems. Their study prompted further work by mathematical researchers and, in turn, benefited from the application of new methods in pure mathematics. It is a vast subject, intimately connected to various sciences such as Physics, Mechanics, Chemistry, Engineering Sciences, with a considerable number of applications to industrial problems.

Although the theory of partial differential equations has undergone a great development in the twentieth century, some fundamental questions remain unresolved. They are essentially concerned with the global existence, regularity and uniqueness of solutions, as well as their asymptotic behavior.

The immediate object of this chapter is to review some improvements achieved in the study of a celebrated nonlinear partial differential system, the incompressible Navier–Stokes equations. The nature of a turbulent motion of a fluid, an ocean for instance, or the creation of a vortex inside it, are two typical problems related to the Navier–Stokes equations, and they are still far from being understood.

From a mathematical viewpoint, one of the most intriguing unresolved questions concerning the Navier–Stokes equations and closely related to turbulence phenomena is the regularity and uniqueness of the solutions to the initial value problem. More precisely, given a smooth datum at time zero, will the solution of the Navier–Stokes equations continue to be smooth and unique for all time? This question was posed in 1934 by Leray [148,149] and is still without answer, neither in the positive nor in the negative. Smale includes the uniqueness and regularity question for the Navier–Stokes equations as one of the 18 open problems of the twentieth century [212].

There is no uniqueness proof except for over small time intervals and it has been questioned whether the Navier–Stokes equations really describe general flows. But there is no proof for nonuniqueness either.

Maybe a mathematical ingenuity is the reason for the missing (expected) uniqueness result. Or maybe the methods used so far are not pertinent and the refractory Navier–Stokes equations should be approached with a different strategy.

Uniqueness of the solutions of the equations of motion is the cornerstone of classical determinism [74]. If more than one solution were associated to the same initial data, the committed determinist will say that the space of the solutions is too large, beyond the real physical possibility, and that uniqueness can be restored if the unphysical solutions are excluded. On the other hand, anarchists will be happy to conclude that the laws of motion are not verified and that chaos reigns. More precisely, a nonuniqueness result would represent such an insulting paradox to classical determinism, that the introduction of a more sophisticated model for the study of the motion of a viscous fluid would certainly be justified [39,42,84,119].

Thirty years ago, Shinbrot wrote [211]:

> Without the d'Alembert (and other paradoxes), who would have thought it necessary to study more intricate models than the ideal fluid? However, it is usually through paradoxes that mathematical

work has the greatest influence on physics. In terms of existence and uniqueness theory, this means that the most important thing to discover is what is not true. When one proves the Navier–Stokes equations have solutions, the physicist yawns. If one can prove these solutions are not unique (say), he opens his eyes instead of his mouth. Thus, when we prove existence theorems, we are only telling the world where paradoxes are not and perhaps sweeping away some of the mist that surrounds the area where they are.

If the problem of uniqueness relates to the predictive power aspect of the theory, the existence issue touches the question of the self-consistency of the physical model involved in the Navier–Stokes equations; if no solution exists, then the theory is empty.

In the nineteenth century, the existence problems arising from mathematical physics were studied with the aim of finding exact solutions to the corresponding equations. This is only possible in particular cases. For instance, very few exact solutions of the Navier–Stokes equations were found and, except for some exact stationary solutions, almost all of them do not involve the specifically nonlinear aspects of the problem, since in general the corresponding nonlinear terms in the Navier–Stokes equations vanish.

In the twentieth century, the strategy changed. Instead of explicit formulas in particular cases, the problems were studied in all their generality. This led to the concept of weak solutions. The price to pay is that only the existence of the solutions can be ensured. In fact, the construction of weak solutions as the limit of a subsequence of approximations leaves open the possibility that there is more than one distinct limit, even for the same sequence of approximations.

The uniqueness question is among the most important unsolved problems in fluid mechanics: "Instant fame awaits the person who answers it. (Especially if the answer is negative!)" [211]. Moreover, as for the solutions of the Navier–Stokes equations, such a uniqueness result is not available for the solutions of the Euler equations of ideal fluids, or the Boltzmann equation of rarefied gases, or the Enskog equation of dense gases either.

A question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the Navier–Stokes equations blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost.

One may imagine that blow-up of initially regular solutions never happens, or that it becomes more likely as the initial norm increases, or that there is blow-up, but only on a very thin set of probability zero. Nobody knows the answer and the Clay Mathematical Institute is offering a prize for it [80]. As Fefferman [80] remarks, finite blow-up in the Euler equation of an "ideal" fluid is an open and challenging mathematical problem as it is for the Navier–Stokes equations. Constantin [67] suggests that it is finite time blow-up in the Euler equations that is the physically more important problem, since blow-up requires large gradients in the limit of zero viscosity. The best result in this direction concerning the possible loss of smoothness for the Navier–Stokes equations was obtained by Caffarelli, Kohn and Nirenberg [31,151], who proved that the one-dimensional Hausdorff measure of the singular set is zero.

After providing such a pessimistic scenery, revealing our lack of comprehension in the study of the Navier–Stokes equations, let us briefly recall here some more encouraging, even if partial, research directions. Roughly speaking, we can summarize the discussion

by saying that if "some quantity" turns out to "be small", then the Navier–Stokes equations are well posed in the sense of Hadamard (existence, uniqueness and stability of the corresponding solutions).

For instance, a unique global solution exists provided the data – the initial value and the exterior force – are small, and the solution is smooth depending on smoothness of the data. Another quantity that can be small is the dimension. If we are in dimension $n = 2$, the situation is easier than in dimension $n = 3$ and completely understood [152,218]. Finally, if the domain $\Omega \subset \mathbb{R}^3$ is small, in the sense that Ω is thin in one direction, say $\Omega = \omega \times (0, \varepsilon)$, then the question is also settled [235].

Other good news is contained in the following pages. They reflect the progress achieved in the last seven years by approaching the Navier–Stokes equations with mathematical tools directly taken from the harmonic analysis world. We mean the use of the Fourier transform and its natural heirs, better suited for the study of nonlinear problems: the Littlewood–Paley decomposition, the paraproducts, the Besov spaces and the wavelets.

Motivated by a somewhat esoteric paper of Federbush entitled "Navier and Stokes meet the wavelets" [78], in 1995 we launched an ambitious program [34]: solve the *nonlinear Navier–Stokes* equations by means of *wavelet transform* and *Besov spaces*. Of course, at the origin of our hopes was the remark that it is possible to solve the *linear heat* equation by *Fourier transform* in *Sobolev spaces*, a very tempting comparison indeed.

Following these ideas and this program, some important results were obtained. They concern the existence of a global solution for highly oscillating data (Section 4), the uniqueness of this solution (Section 5) and its asymptotic behavior, via the existence of self-similar solutions (Section 6).

In the following pages, after recalling these results, we will realize, a posteriori, that the harmonic analysis tools were not necessary at all for their discovery. In fact, each proof of the previous theorem (existence, uniqueness, self-similar solutions) originally found by means of 'Fourier analysis methods', more precisely, by using 'Besov spaces', was followed, shortly after its publication, by a 'real variable methods' proof.

Temam [217] was able to construct a global solution with highly oscillating data by using a classical Sobolev space. This solution was shown to be unique by Meyer [166], with a proof that makes use of a Lorentz space, instead of a Besov one. Finally, Le Jan and Sznitman [138] discovered an elementary space for the existence of self-similar solutions.

The historical details that led to each theorem and each proof are contained in the paper entitled "Viscous flows in Besov spaces" [37], that should be considered as a companion to this article.

1. Preliminaries

1.1. *The Navier–Stokes equations*

We study the Cauchy problem for the Navier–Stokes equations governing the time evolution of the velocity $v(t, x) = (v_1(t, x), v_2(t, x), v_3(t, x))$ and the pressure $p(t, x)$ of an

incompressible viscous fluid (whose viscosity coefficient is given by the positive constant ν) filling all of $\mathbb{R}^3$ and in the presence of an external force $\phi(t, x)$:

$$
\begin{aligned}
&\frac{\partial v}{\partial t} - \nu \Delta v = -(v \cdot \nabla)v - \nabla p + \phi, \\
&\nabla \cdot v = 0, \\
&v(0) = v_0, \quad x \in \mathbb{R}^3, t \geqslant 0.
\end{aligned}
\tag{1}
$$

Here, the external force $\phi(t, x)$ will be considered as arising from a potential $V(t, x)$ in such a way that

$$
\phi = \nabla \cdot V
\tag{2}
$$

which means, that

$$
\phi_j = \sum_{k=1}^{3} \partial_k V_{kj}, \quad j = 1, 2, 3.
\tag{3}
$$

As we will describe in Section 6.4, more general types of forces can be considered, this is done for instance in the recent paper [42,40] (for other examples see also [130,133]).

We will also assume that the viscosity ν is equal to one. This can be done, without loss of generality, because of the invariant structure of the Navier–Stokes equations and we will return to this issue in Section 3.2.

Finally, thanks to the divergence-free property $\nabla \cdot v = 0$, expressing the incompressibility of the fluid, we can write $(v \cdot \nabla)v = \nabla \cdot (v \otimes v)$. This remark is important because the product of two tempered distributions is not always defined, whereas it is always possible to take the derivative (in the distribution sense) of an L^1_{loc} function. Thus, it will be enough to require $v \in L^2_{\text{loc}}$ in order to make the quadratic term $\nabla \cdot (v \otimes v)$ well defined.

Here and in the following, we say that a vector $a = (a_1, a_2, a_3)$ belongs to a function space X if $a_j \in X$ holds for every $j = 1, 2, 3$, and we put $\|a\| = \max_{1 \leqslant j \leqslant 3} \|a_j\|$. To be more precise, we should write $X(\mathbb{R}^3)$ instead of X (for instance $v = (v_1, v_2, v_3) \in L^2_{\text{loc}}$ means $v_j \in L^2_{\text{loc}}(\mathbb{R}^3)$ for every $j = 1, 2, 3$). In order to avoid any confusion, if the space is not $\mathbb{R}^3$ (for example, if the dimension is 2 or if the space is a bounded domain Ω_b as considered at the end of Section 5.1) we will write it explicitly (say $X(\mathbb{R}^2)$ or $X(\Omega_b)$). The reason why we are mainly interested in the whole space $\mathbb{R}^3$ (or more generally $\mathbb{R}^n$, $n \geqslant 2$) is that we will make constant use of Fourier transform tools, that are easier to handle in the case of the whole space (or a bounded space with periodic conditions, as in [222]) than that of a domain with boundaries. A detailed analysis of the problems that can occur if the Navier–Stokes (or more general) equations are supplemented by the homogeneous Dirichlet (no-slip) boundary conditions is contained in [83].

Our attention will be focused on the existence of solutions $v(t, x)$ to (1) in the space $\mathcal{C}([0, T); X)$ that are strongly continuous functions of $t \in [0, T)$ with values in the Banach space X of vector distributions. Depending on whether T will be *finite* ($T < \infty$) or *infinite* ($T = \infty$) we will obtain respectively *local* or *global* (in time) solutions.

Before introducing the appropriate functional setting, let us transform the system (1) into the operator equation [30,87,117]:

$$\frac{\mathrm{d}v}{\mathrm{d}t} - Av = -\mathbb{P}\nabla \cdot (v \otimes v) + \mathbb{P}\phi,$$

$$v(0) = v_0, \quad x \in \mathbb{R}^3, t \geqslant 0,$$

(4)

where A is formally defined as the operator $A = -\mathbb{P}\Delta$ and $\mathbb{P}$ is the Leray–Hopf orthogonal projection operator onto the divergence-free vector field defined as follows.

We let

$$D_j = -\mathrm{i}\frac{\partial}{\partial x_j}, \quad j = 1, 2, 3; \mathrm{i}^2 = -1,$$

(5)

and we denote the Riesz transforms by

$$R_j = D_j(-\Delta)^{-1/2}, \quad j = 1, 2, 3.$$

(6)

For an arbitrary vector field $v(x) = (v_1(x), v_2(x), v_3(x))$ on $\mathbb{R}^3$, we set

$$z(x) = \sum_{k=1}^{3} (R_k v_k)(x)$$

(7)

and define the Leray–Hopf operator $\mathbb{P}$ by

$$(\mathbb{P}v)_j(x) = v_j(x) - (R_j z)(x) = \sum_{k=1}^{3} (\delta_{jk} - R_j R_k)v_k, \quad j = 1, 2, 3.$$

(8)

Another equivalent way to define $\mathbb{P}$ is to make use of the properties of the Fourier transform and write

$$(\widehat{\mathbb{P}v})_j(\xi) = \sum_{k=1}^{3} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \hat{v}_k(\xi), \quad j = 1, 2, 3.$$

(9)

As such, $\mathbb{P}$ is a pseudo-differential operator of degree zero and is an orthogonal projection onto the kernel of the divergence operator. In other words the pressure p in (1) ensures that the incompressibility condition $\nabla \cdot v = 0$ is satisfied.

Finally, making use of this projection operator $\mathbb{P}$ and the semigroup

$$S(t) = \exp(-tA),$$

(10)

it is a straightforward procedure to reduce the operator equation (4) into the following

integral equation

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (v \otimes v)(s)\,ds + \int_0^t S(t-s)\mathbb{P}\nabla \cdot V(s)\,ds. \tag{11}$$

On purpose, we are being a little cavalier here: we shall not justify the formal transition $(1) \to (4) \to (11)$. We shall rather start from (11) and prove the existence and uniqueness of a solution $v(t, x)$ for it. Then, we shall prove that this solution is regular enough to form, with an appropriate pressure $p(t, x)$, a classical solution of the system (1).

Since our attention will essentially be devoted to the study of the integral equation (11) and since we will only consider the case of the all space $\mathbb{R}^3$, so that the semigroup $S(t)$ reduces to the well-known heat semigroup $\exp(t\Delta)$, we will separate the different contributions in (11) in the following way: the linear term containing the initial data

$$S(t)v_0 =: \exp(t\Delta)v_0, \tag{12}$$

the bilinear operator expressing the nonlinearity of the equation

$$B(v, u)(t) =: -\int_0^t \exp\big((t-s)\Delta\big)\mathbb{P}\nabla \cdot (v \otimes u)(s)\,ds \tag{13}$$

and finally the linear operator L involving the external force

$$L(V)(t) =: \int_0^t \exp\big((t-s)\Delta\big)\mathbb{P}\nabla \cdot V(s)\,ds. \tag{14}$$

The precise meaning of the integral defined by (13) in different function spaces is one of the main problems arising from this approach and will be discussed carefully in the following section.

Let us note here that there is a kind of competition in this integral term between the regularizing effect represented by the heat semigroup $S(t-s)$ and the loss of regularity that comes from the differential operator ∇ and from the pointwise multiplication $v \otimes u$. This loss of regularity is illustrated by the following simple example: if two (scalar) functions f and g are in H^1, their product only belongs to $H^{1/2}$ and their derivative $\partial(fg)$ is even less regular as it belongs to $H^{-1/2}$.

1.2. *Classical, mild and weak solutions*

As yet the existence of a global solution in time has not been proved nor disproved for a three-dimensional flow and sufficiently general initial conditions; but as we will see in the following pages, a global, regular solution does exist whenever the initial data are highly oscillating or sufficiently small in certain function spaces.

To begin with, it is necessary to clarify the meaning of "solution of the Navier–Stokes equations", because, since the appearance of the pioneer papers of Leray, the word "solution" has been used in a more or less generalized sense. Roughly speaking, two main types of solutions can be distinguished: "strong solutions" (for which existence and uniqueness are known) and "weak solutions" (for which only the existence is known).

In the following pages, we will take the term "solution" in the generic sense of classical ordinary differential equations in t with values in the space of tempered distributions $\mathcal{S}'$, in order to be able to use the Fourier transforms tools. This interpretation is suggested by the notion of solution in the sense of distribution used in evolution equations. Moreover, we will ask that the function space X, to which the initial data v_0 belong, is such that $X \hookrightarrow L_{\text{loc}}^2$, in order to be able to give a (distributional) meaning to the nonlinear term $(v \cdot \nabla)v = \nabla \cdot (v \otimes v)$. More generally, we will ask $v \in L_{\text{loc}}^2([0, T); \mathbb{R}^3)$.

In the recent papers of Amann [1] and of Lemarié [142,145], we can count many different definitions of solutions (see also [71]) distinguished only by the class of functions to which they are supposed to belong: *classical*, *strong*, *mild*, *weak*, *very weak*, *uniform weak* and *local Leray* solutions of the Navier–Stokes equations!

We will not present all the possible definitions here but concentrate our attention on three cases, respectively classical (Hadamard), weak (Leray) and mild (Yosida) solutions.

DEFINITION 1 (Classical). A classical solution $(v(t, x), p(t, x))$ of the Navier–Stokes equations is a pair of functions $v : t \to v(t)$ and $p : t \to p(t)$ satisfying the system (1), for which all the terms appearing in the equations are continuous functions of their arguments. More precisely, a classical solution is a solution to the system (1) that verifies:

$$v(t, x) \in \mathcal{C}\big([0, T); E\big) \cap \mathcal{C}^1\big([0, T); F\big), \tag{15}$$

$$E \hookrightarrow F \quad \text{(continuous embedding)}, \tag{16}$$

$$v \in E \quad \Longrightarrow \quad \Delta v \in F \quad \text{(continuous operator)}, \tag{17}$$

$$v \in E \quad \Longrightarrow \quad \nabla \cdot (v \otimes v) \in F \quad \text{(continuous operator)}, \tag{18}$$

where E and F are two Banach spaces of distributions.

For example, if E is the Sobolev space H^s and $s > 3/2$ (thus giving H^s the structure of an algebra when endowed with the usual product of functions), we can chose $F = H^{s-2}$, because $\Delta v \in H^{s-2}$ and $\nabla \cdot (v \otimes v) \in H^{s-1} \hookrightarrow H^{s-2}$.

As we recalled in the Introduction, it is very difficult to ensure the existence of classical solutions, unless we look for exact solutions (that do not involve the specific aspects of the problem, since in general the corresponding nonlinear terms in the equations vanish), or we impose very restrictive conditions on the initial data (see Section 3). This is not the case when we take the word solution in the weak sense given by Leray.

DEFINITION 2 (Weak). A *weak solution* $v(t, x)$ of the Navier–Stokes equations in the sense of Leray and Hopf is supposed to have the following properties:

$$v(t, x) \in L^\infty\big([0, T); \mathbb{P}L^2\big) \cap L^2\big([0, T); \mathbb{P}H^1\big) \tag{19}$$

and

$$\int_0^T \left(-\langle v, \partial_t \varphi \rangle + \langle \nabla v, \nabla \varphi \rangle + \langle (v \cdot \nabla) v, \varphi \rangle \right) \mathrm{d}s = \langle v_0, \varphi(0) \rangle + \int_0^T \langle \phi, \varphi \rangle \, \mathrm{d}s \tag{20}$$

for any $\varphi \in \mathcal{D}([0, T); \mathbb{P}\mathcal{D})$. The symbol $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product, whereas $\mathbb{P}X$ denotes the subspace of X (here $X = L^2$, H^1 or $\mathcal{D}$) of solenoidal functions,[1] characterized by the divergence-free condition $\nabla \cdot v = 0$. Finally, such a weak solution is supposed to verify the following energy inequality

$$\frac{1}{2} \|v(t)\|_2^2 + \int_0^t \|\nabla v(s)\|_2^2 \, \mathrm{d}s \leqslant \frac{1}{2} \|v(0)\|_2^2 + \int_0^t \langle \phi, v \rangle \, \mathrm{d}s, \quad t > 0. \tag{21}$$

Sometimes this inequality is satisfied not only on the interval $(0, t)$ but on all intervals $(t_0, t_1) \subset (0, T)$ except possibly for a set of measure zero. Such a solution is called *turbulent* in Leray's papers.

Finally, after the papers of Kato and his collaborators, we got used to calling *mild solutions* a third category of solutions, whose existence is obtained by a fixed point algorithm applied to the integral equation (11). In other words, the Navier–Stokes equations are studied by means of semigroup techniques as in the pioneering papers of Yosida [238]. More precisely, mild solutions are defined in the following way.

DEFINITION 3 (Mild). A *mild solution* $v(t, x)$ of the Navier–Stokes equations satisfies the integral equation (11) and is such that

$$v(t, x) \in \mathcal{C}\big([0, T); \mathbb{P}X\big), \tag{22}$$

where X is a Banach space of distributions on which the heat semigroup $\{\exp(t\Delta); t \geqslant 0\}$ is strongly continuous and the integrals in (11) are well defined in the sense of Bochner.

Historically, the introduction of the term "mild" in connection with the integral formulation for the study of an arbitrary evolution equation goes back to Browder [30]. We do not expect to use the energy inequality, but we hope to ensure in this way the uniqueness of the solution, in other words that the solution is strong. This is in contrast with Leray's construction of *weak solutions*, relying on compactness arguments and *a priori* energy estimates. Moreover, the fixed point algorithm is stable and constructive. Thus the problem of defining mild solutions is closely akin to the question of knowing whether the Cauchy problem for Navier–Stokes equations is well posed in the sense of Hadamard. This question will be discussed in Section 7 in connection with the theory of stability and Lyapunov functions.

Let us recall that for a function $u(t, \cdot)$ that takes values in a Banach space E, the integral $\int_0^T u(t, \cdot) \, \mathrm{d}t$ exists either because $\int_0^T \|u(t, \cdot)\|_E \, \mathrm{d}t < \infty$ (in this case we say that the

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[1] In the literature this space is usually denoted by X_σ.

integral is defined in the sense of Bochner) or because $\int_0^T |\langle u(t, \cdot), y \rangle| \, dt$ converges for any vector y of the dual (or pre-dual) E' of E (the integral is said to be weakly convergent). The weak convergence is ensured by the oscillatory behavior of $u(t, \cdot)$ in the Banach space E.

Now, the oscillatory property of the bilinear term arising from the Navier–Stokes equations is systematically taken into account in all papers that are based on the energy inequality, in particular $\langle B(v, v), v \rangle = 0$ as long as $\nabla \cdot v = 0$. In the following pages, we will *never* take advantage of this remarkable property, for we will only consider functional spaces where it is *not* possible to write $\langle B(v, v), v \rangle$. In fact, $B(v, v)$ will *never* belong to a space that is a dual of the one to which v belongs. This is the reason why our works ([46,47] excepted) are *not* based on the innermost structure of the Navier–Stokes equations and can be easily extended to other nonlinear partial differential equations [14–17,52–54, 89,90,113,156,183,184,190–192,196–202,222,223].

More explicitly, in the literature concerning the existence and uniqueness of mild solutions for the Navier–Stokes equations as inaugurated by Fujita and Kato's celebrated papers [87,117], the oscillatory behavior of $B(v, u)$ is lost from the very beginning because, by definition, mild solutions require *strong* estimates in the *strong* topology, so that $B(v, u)$ can be replaced by $|B(v, u)|$ without affecting the corresponding existence and uniqueness results.

On the other hand, as far as the *weak* solutions are concerned, introduced in the pioneering papers by Leray [148–150], the oscillatory behavior of $B(v, u)$ is frequently analyzed by means of the well-known identity

$$\langle \nabla \cdot (u \otimes v), v \rangle = 0, \tag{23}$$

where $\nabla \cdot u = 0$. In that case the problem is different, for the above identity does not allow a great flexibility in the choice of the functional setting, that is forced to be defined in terms of an energy norm (e.g., L^2, H^1, ...).

1.3. *Navier meets Fourier*

The title of this section is borrowed from a paper by Federbush "Navier and Stokes meet the wavelets" [78,79] that will be dealt with in Section 2.4.

The Navier–Stokes equations did not yet exist when Fourier gave the explicit solution of the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= f, \\ u(0) &= u_0. \end{aligned} \tag{24}$$

This equation, governing the evolution of temperature $u(x, t)$, in the presence of an exterior source of heat $f(x, t)$, at a point x and time t of a body assumed here to fill the

whole space $\mathbb{R}^3$, becomes, when we consider its partial Fourier transform with respect to x, an ordinary differential equation in t, whose solution is given by

$$u(t, x) = S(t)u_0 + \int_0^t S(t-s) f(s) \, ds, \tag{25}$$

$S(t)$ being the convolution operator defined as in (12) by the heat semigroup

$$S(t) = \exp(t\Delta) = \left(\frac{1}{4\pi t}\right)^{3/2} \exp\left(-\frac{|x|^2}{4t}\right). \tag{26}$$

The Navier–Stokes equations, that describe the motion of a viscous fluid, were introduced by Navier in 1822 [178], the same year that, by a curious coincidence, Fourier published the celebrated treatise "Théorie analytique de la chaleur" [86], in which he developed in a systematic way the ideas contained in a paper of 1807.

But this is not only a mere coincidence. In fact Navier, engineer of the *Ecole Nationale des Ponts et Chaussées*, was also a very close friend of many mathematicians, in particular Fourier. Fourier had a strong influence on Navier's life and career, both as a friend and as a teacher. In turn, Navier was a noticeable proponent of the important mathematical techniques developed by Fourier.[2]

In this section we want to show how to take advantage of the Fourier transform in order to study the Navier–Stokes equations.

We have already remarked that, following Fourier's method to solve the Navier–Stokes equations for a viscous incompressible fluid, we obtain the integral equation (11), very similar to (25), that led to the concept of a mild equation and a mild solution.

If we want to make use of the Fourier transform again, the second idea that comes to mind is to rewrite (11) componentwise ($j = 1, 2, 3$) in Fourier variables

$$\hat{v}_j(\xi) = \exp\left(-t|\xi|^2\right)\hat{v}_{0j}$$

$$- \int_0^t \exp\left(-(t-s)|\xi|^2\right) \sum_{l,k=1}^3 \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\right)(i\xi_l)\hat{v}_l(\xi) * \hat{v}_k(\xi)$$

$$+ \int_0^t \exp\left(-(t-s)|\xi|^2\right) \sum_{l,k=1}^3 \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\right)(i\xi_l)\widehat{V_{lk}}(\xi) \, ds.$$

We use the notations introduced by Miyakawa in [173] and denote by $F(t, x)$ the tensor kernel associated with the operator $\exp(t\Delta)\mathbb{P}\nabla\cdot$, say

$$\widehat{F_{l,k,j}}(t, \xi) = \exp\left(-t|\xi|^2\right)\left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\right)i\xi_l. \tag{27}$$

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[2]This was not the case for most other engineers of his period. Navier's interests in more mathematical aspects of physics, mechanics and engineers sciences were so deep that, when the suspension bridge across the Seine he had designed collapsed, sarcastic articles appeared in the press against Navier, who was referred to as "that eminent man of science whose calculations fail in Paris" (see [38]).

It is easy to see that the kernel $F(t, x) = \{F_{l,k,j}(t, x)\}$ defined in this way verifies

$$\big|F(t, x)\big| \lesssim |x|^{-\alpha} t^{-\beta/2}, \quad \alpha \geqslant 0, \beta \geqslant 0, \alpha + \beta = 4, \tag{28}$$

and

$$\big\|F(t, x)\big\|_p \lesssim t^{-(4-3/p)/2}, \quad 1 \leqslant p \leqslant \infty. \tag{29}$$

In the following pages we will not take advantage of these general estimates. In fact, we will never use the full structure of the operator $\exp(t\Delta)\mathbb{P}\nabla\cdot$ and our analysis will apply to a more general class of evolution equations.

Let us be more explicit. Our existence and uniqueness theorems for the mild Navier–Stokes equations will be obtained by using the Banach fixed point theorem. The continuity of the bilinear term B as well as the continuity of the linear term L defined in (13) and (14) will be the main ingredients of the proofs. The functional spaces where the initial data will be considered are such that the Riesz transforms operate continuously. The conclusion is easy: we will get rid of the Riesz transforms from the very beginning and limit ourselves to the study of a simplified version of the operator $\exp(t\Delta)\mathbb{P}\nabla\cdot$ giving rise to simplified versions of the operators B and L.

We denote with the letters B_s and L_s these operators defined by

$$B_s(f, g)(t) =: -\int_0^t \big[S(t - s)\dot\Lambda\big](fg)(s)\, ds \tag{30}$$

and

$$L_s(h)(t) =: \int_0^t \big[S(t - s)\dot\Lambda\big]h(s)\, ds, \tag{31}$$

where $f = f(t, x)$, $g = g(t, x)$ and $h(t, x)$ are generic *scalar* fields and

$$\dot\Lambda =: \sqrt{-\Delta} \tag{32}$$

denotes the well-known Calderón's homogeneous pseudo-differential operator whose symbol in Fourier transform is $|\xi|$.

In order to obtain such simplified *scalar* versions of the operators B and L, we have not taken into account all Riesz transforms contained in the full *vectorial* operators. For example, as far as the continuity of the bilinear operator is concerned in a certain function space, we can pass from the full vectorial operator B,

$$j \in \{1, 2, 3\},$$

$$B(u, v)_j = -\mathrm{i} \sum_{m=1}^3 R_m B_s(u_m, v_j) + \mathrm{i} \sum_{k=1}^3 \sum_{l=1}^3 R_j R_k R_l B_s(u_l, v_k), \tag{33}$$

to its scalar simplified version B_s just by using the continuity of the Riesz transforms in this space.

With this simplification in mind, and by recalling the elementary properties of the Fourier transform, we finally get an even simpler expression for the bilinear term (that by abuse of notation will be always denoted by the letter B):

$$B(f, g) = - \int_0^t (t - s)^{-2} \Theta\left(\frac{\cdot}{\sqrt{t - s}}\right) * (fg)(s) \, ds, \tag{34}$$

where $f = f(t, x)$ and $g = g(t, x)$ are two scalar fields and $\Theta = \Theta(x)$ is a function of x whose Fourier transform is given by

$$\widehat{\Theta}(\xi) = |\xi| e^{-|\xi|^2}. \tag{35}$$

As such, Θ is analytic, behaves like $O(|x|^{-4})$ at infinity (this can also be deduced by (28) for $\alpha = 4$ and $\beta = 0$) and its integral is zero.

In the same way, the linear operator L involving the external force will be treated in the simplified scalar form

$$L(h) = \int_0^t (t - s)^{-2} \Theta\left(\frac{\cdot}{\sqrt{t - s}}\right) * h(s) \, ds. \tag{36}$$

In particular, we notice that

$$B(f, g) = -L(fg) \tag{37}$$

which allows to treat both the bilinear and the linear terms in exactly the same way. This is why, for the sake of simplicity, in the following pages we will only consider the case when there is no external force and refer the reader to [39,42,40,47] for the general case.

2. Functional setting of the equations

2.1. *The Littlewood–Paley decomposition*

Let us start with the Littlewood–Paley decomposition in $\mathbb{R}^3$. To this end, we take an arbitrary function φ in the Schwartz class $\mathcal{S}$ and whose Fourier transform $\hat{\varphi}$ is such that

$$0 \leqslant \hat{\varphi}(\xi) \leqslant 1, \quad \hat{\varphi}(\xi) = 1 \quad \text{if } |\xi| \leqslant \frac{3}{4}, \qquad \hat{\varphi}(\xi) = 0 \quad \text{if } |\xi| \geqslant \frac{3}{2}, \tag{38}$$

and let

$$\psi(x) = 8\varphi(2x) - \varphi(x), \tag{39}$$

$$\varphi_j = 2^{3j}\varphi(2^j x), \qquad j \in \mathbb{Z}, \tag{40}$$

$$\psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}. \tag{41}$$

We denote by S_j and Δ_j, respectively, the convolution operators with φ_j and ψ_j. Finally, the set $\{S_j, \Delta_j\}_{j \in \mathbb{Z}}$ is the Littlewood–Paley decomposition, so that

$$I = S_0 + \sum_{j \geqslant 0} \Delta_j. \tag{42}$$

To be more precise, we should say "a decomposition", because there are different possible (equivalent) choices for the function φ. On the other hand, for an arbitrary tempered distribution f, the last identity gives

$$f = \lim_{j \to \infty} S_0 f + \sum_{j \geqslant 0} \Delta_j f. \tag{43}$$

The interest in decomposing a tempered distribution into a sum of dyadic blocks $\Delta_j f$, whose support in Fourier space is localized in a corona, comes from the nice behavior of these blocks with respect to differential operations. This fact is illustrated by the following celebrated Bernstein's lemma in $\mathbb{R}^3$, whose proof can be found in [162].

LEMMA 1. *Let* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $k \in \mathbb{N}$, *then one has*

$$\sup_{|\alpha| = k} \left\| \partial^\alpha f \right\|_p \simeq R^k \| f \|_p \tag{44}$$

and

$$\| f \|_q \lesssim R^{3(1/p - 1/q)} \| f \|_p \tag{45}$$

whenever f *is a tempered distribution in* $\mathcal{S}'$ *whose Fourier transform* $\hat{f}(\xi)$ *is supported in the corona* $|\xi| \simeq R$.

In the case of a function whose support is a ball (as, for instance, for $S_j f$) the lemma reads as follows:

LEMMA 2. *Let* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $k \in \mathbb{N}$, *then one has*

$$\sup_{|\alpha| = k} \left\| \partial^\alpha f \right\|_p \lesssim R^k \| f \|_p \tag{46}$$

and

$$\| f \|_q \lesssim R^{3(1/p - 1/q)} \| f \|_p \tag{47}$$

whenever f *is a tempered distribution in* $\mathcal{S}'$ *whose Fourier transform* $\hat{f}(\xi)$ *is supported in the ball* $|\xi| \lesssim R$.

Let us go back to the decomposition of the unity (42) and (43). It was introduced in the early 1930s by Littlewood and Paley to estimate the L^p norm of trigonometric Fourier series when $1 < p < \infty$. If we omit the trivial case $p = 2$, it is not possible to ensure the belonging of a generic Fourier series to the Lebesgue space L^p by simply using its Fourier coefficients, but this becomes true if we consider instead its dyadic blocks. In the case of a function f (not necessarily periodic), this property is given by the following equivalence

$$\text{if } 1 < p < \infty \quad \text{then } \|f\|_p \simeq \|S_0 f\|_p + \left\| \left(\sum_{j=0}^{\infty} |\Delta_j f(\cdot)|^2 \right)^{1/2} \right\|_p. \tag{48}$$

It is even easier to prove that the classical Sobolev spaces $H^s = H_2^s$, $s \in \mathbb{R}$, can be characterized by the following equivalent norms

$$\|f\|_{H^s} \simeq \|S_0 f\|_2 + \left(\sum_{j=0}^{\infty} 2^{2js} \|\Delta_j f\|_2^2 \right)^{1/2}. \tag{49}$$

As far as the more general norms $\|f\|_{H_p^s} = \|(I - \Delta)^{s/2} f\|_p$, $s \in \mathbb{R}$, $1 < p < \infty$, corresponding to the Sobolev–Bessel spaces H_p^s (that is, when s is an integer, reduce to the well-known Sobolev spaces $W^{s,p}$ whose norm are given by $\|f\|_{W_p^s} = \sum_{|\alpha| \leqslant s} \|\partial^\alpha f\|_p$) we will see in the next section how (49) has to be modified.

Another easier case we wish to present here is provided by the Hölder–Zygmund spaces $\mathcal{C}^s$, $s \in \mathbb{R}$, that can be characterized by the following norms

$$\|f\|_{\mathcal{C}^s} \simeq \|S_0 f\|_\infty + \sup_{j \geqslant 0} 2^{js} \|\Delta_j f\|_\infty. \tag{50}$$

We will not prove this property here and we refer the reader to [82]. Let us just remind the reader of the usual definition of these spaces, in order to better appreciate the simplicity of (50). If $0 < s < 1$ we denote the Hölder space by

$$\|f\|_{\mathcal{C}^s} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s}. \tag{51}$$

As it is well known, this definition has to be modified in the case $s = 1$ in the following way

$$\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x + y) + f(x - y) - 2f(x)|}{|x - y|} \tag{52}$$

and defines the Zygmund class $\mathcal{C}^1$. It is now easy to define, for any $s > 0$, the quantities

$$\|f\|_{\mathcal{C}^s} = \|f\|_\infty + \sum_{i=1}^{n} \|\partial_i f\|_{\mathcal{C}^{s-1}}. \tag{53}$$

In the case $s < 0$ we define the Hölder–Zygmund spaces by the following rule:

$$\mathcal{C}^{s-1} = \left\{ f = \sum_{i=1}^{n} \partial_i g_i, \, g_i \in \mathcal{C}^s \right\},$$

$$\|f\|_{\mathcal{C}^{s-1}} = \inf_{i=1,2,\dots,n} \sup \|g_i\|_{\mathcal{C}^s}, \tag{54}$$

the infimum being taken over the set of g_i such that $f = \sum_{i=1}^{n} \partial_i g_i$.

Before defining the Besov spaces that will play a key role in our study of the Navier–Stokes equations, let us recall the homogeneous decomposition of the unity, analogous to (42), but containing also all the low frequencies ($j < 0$), say

$$I = \sum_{j \in \mathbb{Z}} \Delta_j. \tag{55}$$

If we apply this identity to an arbitrary tempered distribution f, we may be tempted to write

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f, \tag{56}$$

but, at variance with (43), this identity has no meaning in $\mathcal{S}'$ for several reasons. First of all, the sum in (56) does not necessarily converge in $\mathcal{S}'$ as we can see if we consider a test function $g \in \mathcal{S}$ whose Fourier transform is equal to 1 near the origin, because in this case the quantity $\langle \Delta_j f, g \rangle$ is, for all $j \ll 0$, a positive constant not depending on j. And, even when the sum is convergent, the convergence has to be understood modulo polynomials, because, for these particular functions P, we have $\Delta_j P = 0$ for all $j \in \mathbb{Z}$.

A way to restore the convergence is to "sufficiently" derive the formal series $\sum_{j \in \mathbb{Z}}$ as it stated in the following lemma (see [21,22,183] for a simple proof).

LEMMA 3. *For any tempered distribution f, there exists an integer d such that, for any α, $|\alpha| \geqslant d$, the series $\sum_{j<0} \partial^\alpha (\Delta_j f)$ converges in $\mathcal{S}'$.*

The following corollary, whose proof follows from the previous lemma, gives the correct meaning to the convergence (56), that is modulo polynomials.

COROLLARY 1. *For any integer N, there exists a polynomial P_N of degree $< d$ such that the quantity $\sum_{j=-N}^{\infty} \Delta_j f - P_N$ converges in $\mathcal{S}'$ when $N \to \infty$.*

In such a way, the series $\Delta_j f$ is always well defined; furthermore, it is not difficult to prove that the difference $f - \sum_{j \in \mathbb{Z}} \Delta_j f$ has its spectrum reduced to zero; in other words, it is a polynomial. In this way, the convergence in (56), that fails to be valid in $\mathcal{S}'$, is ensured in the quotient space $\mathcal{S}'/\mathcal{P}$.

2.2. *The Besov spaces*

The Littlewood–Paley decomposition is very useful because we can define (independently of the choice of the initial function φ) the following (inhomogeneous) Besov spaces [82,185].

DEFINITION 4. Let $0 < p, q \leqslant \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (inhomogeneous) Besov space $B_q^{s,p}$ if and only if

$$\|S_0 f\|_q + \left(\sum_{j>0} \left(2^{sj} \|\Delta_j f\|_q \right)^p \right)^{1/p} < \infty. \tag{57}$$

For the sake of completeness, we also define the (inhomogeneous) Triebel–Lizorkin spaces, even if we will not make a great use of them in the study of the Navier–Stokes equations.

DEFINITION 5. Let $0 < p \leqslant \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (inhomogeneous) Triebel–Lizorkin space $F_q^{s,p}$ if and only if

$$\|S_0 f\|_q + \left\| \left(\sum_{j>0} \left(2^{sj} |\Delta_j f| \right)^p \right)^{1/p} \right\|_q < \infty. \tag{58}$$

It is easy to see that the above quantities define a norm if $p, q \geqslant 1$ and a quasi-norm in general, with the usual convention that $p = \infty$ in both cases corresponds to the usual L^∞ norm. On the other hand, we have not included the case $q = \infty$ in the second definition because the L^∞ norm has to be replaced here by a more complicated Carleson measure (see [82]).

As we have already remarked before for some particular values of s, p, q, see (48)–(50), the Besov and Triebel–Lizorkin spaces generalize the usual Lebesgue ones, for instance,

$$L^q = F_q^{0,2}, \quad 1 < q < \infty, \tag{59}$$

and more generally the Sobolev–Bessel spaces,

$$H_q^s = F_q^{s,2}, \quad s \in \mathbb{R}, 1 < q < \infty, \tag{60}$$

and the Hölder ones,

$$\mathcal{C}^s = B_\infty^{s,\infty}. \tag{61}$$

Another interesting case is given by the space $F_q^{0,2}$ with $0 < q \leqslant 1$ that corresponds to a local version of the Hardy space, whereas $F_\infty^{0,2}$ gives the local version *bmo* of the John and Nirenberg space *BMO* of Bounded Mean Oscillation functions[3] whose norm is defined by

$$\| f \|_{BMO} = \sup_B \left(\frac{1}{\mu(B)} \int_B |f - f_B|^2 \, \mathrm{d}x \right)^{1/2}, \tag{62}$$

where B stands for the set of Euclidean balls, $\mu(B)$ the volume of B and f_B denotes the average of the function f over B, say $f_B = \frac{1}{|B|} \int_B f(x) \, \mathrm{d}x$. It is clear that this quantity is in general a seminorm, unless we argue modulo constant functions (whose *BMO*-norm is zero). Moreover, it is evident that $L^\infty \hookrightarrow BMO$ but these spaces are different, because the functions $f(x) = \ln |p(x)|$, for all polynomials $p(x)$, belong to *BMO* but not to L^∞.

A space that will be useful in the following pages is provided by the set of functions which are derivatives of functions in *BMO*. More precisely, we are talking about the space introduced by Koch and Tataru in [123], that is denoted by BMO^{-1} (or by ∇BMO) and is defined as the space of tempered distributions f such that there exists a vector function $g = (g_1, g_2, g_3)$ belonging to *BMO* such that

$$f = \nabla \cdot g. \tag{63}$$

The norm in BMO^{-1} is defined by

$$\| f \|_{BMO^{-1}} = \inf_{g \in BMO} \sum_{j=1}^3 \| g_j \|_{BMO}. \tag{64}$$

At this point, in order to provide the reader with the dyadic decomposition of the classical Hardy $\mathcal{H}^q$, *BMO* and BMO^{-1} spaces, we have to recall that their norms, at variance with the local ones, are "homogeneous".

Let us be more explicit and consider some familiar examples. The Lebesgue space L^p is "homogeneous", because its norm satisfies, with respect to the dilatation group, the following invariance $\| f(\lambda \cdot) \|_p = \lambda^{-3/p} \| f \|_p$ for all $\lambda > 0$. On the other hand, the Sobolev space H^1 normed with $\| f \|_{H^1} = \| f \|_2 + \| \nabla f \|_2$ does not verify a property of this type because the two terms composing the norm have different homogeneity (resp. $\lambda^{-3/2}$ and $\lambda^{1-3/2}$). A possible way to restore the scaling invariance would be to forget the L^2 part and define the "homogeneous" Sobolev space $\dot{H}^1$ simply by $\| f \|_{\dot{H}^1} = \| \nabla f \|_2$. Of course the attentive reader, armed with the discussion that follows (56), will protest that this quantity is not a norm, unless we work in $\mathcal{S}'$ modulo polynomials (in the case of $\dot{H}^1$, modulo constants would be sufficient). A very simple condition that prevents constant functions to belong to $\dot{H}^1$ is given by [166]:

$$\int_{|x| \leqslant R} |f(x)| \, \mathrm{d}x = \mathrm{o}\big(R^3\big), \quad R \to +\infty. \tag{65}$$

---

[3]For a different interpretation of the acronym... see [185], page 175!

A stronger, but more natural condition is provided by the celebrated Sobolev embedding in $\mathbb{R}^3$

$$\|f\|_6 \lesssim \|\nabla f\|_2, \tag{66}$$

thus suggesting the following definition: A function f belongs to $\dot{H}^1$ if and only if ∇f belongs to L^2 and f belongs to L^6, the norm of f in $\dot{H}^1$ being $\|\nabla f\|_2$. Indeed, this definition is equivalent to defining $\dot{H}^1$ as the closure of the test functions space C_0^∞ for the norm $\|f\|_{\dot{H}^1} = \|\nabla f\|_2$. In the same way, we define the space $\dot{H}_p^s$ when $s < 3/p$ as the closure of the space

$$\mathcal{S}_0 = \left\{ f \in \mathcal{S}, 0 \notin \operatorname{Supp} \hat{f} \right\} \tag{67}$$

for the norm

$$\|f\|_{\dot{H}_p^s} = \left\| \dot{\Lambda}^s f \right\|_p, \tag{68}$$

where, as usual, $\dot{\Lambda} = \sqrt{-\Delta}$ denotes the homogeneous Calderón pseudo-differential operator (see Section 1.3). Finally, when $3/p + d \leqslant s < 3/p + d + 1$ and d is an integer, $\dot{H}_p^s$ is a space of distributions modulo polynomials of degree $\leqslant d$.

We are now ready to define the homogeneous version of the Besov and Triebel–Lizorkin spaces [21,22,82,185].

If $m \in \mathbb{Z}$, we denote by $\mathcal{P}_m$ the set of polynomials of degree $\leqslant m$ with the convention that $\mathcal{P}_m = \emptyset$ if $m < 0$. If $q = 1$ and $s - 3/p \in \mathbb{Z}$, we put $m = s - 3/p - 1$; if not, we put $m = [s - 3/p]$, the brackets denoting the integer part function.

DEFINITION 6. Let $0 < p, q \leqslant \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (homogeneous) Besov space $\dot{B}_q^{s,p}$ if and only if

$$\left(\sum_{j \in \mathbb{Z}} \left(2^{sj} \|\Delta_j f\|_q \right)^p \right)^{1/p} < \infty \quad \text{and} \quad f = \sum_{-\infty}^{\infty} \Delta_j f \quad \text{in } \mathcal{S}'/\mathcal{P}_m. \tag{69}$$

DEFINITION 7. Let $0 < p \leqslant \infty$, $0 < q < \infty$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to the (homogeneous) Triebel–Lizorkin space $\dot{F}_q^{s,p}$ if and only if

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(2^{sj} |\Delta_j f| \right)^p \right)^{1/p} \right\|_q < \infty \quad \text{and} \quad f = \sum_{-\infty}^{\infty} \Delta_j f \quad \text{in } \mathcal{S}'/\mathcal{P}_m, \tag{70}$$

with an analogous modification as in the inhomogeneous case when $q = \infty$.

As expected, we have the following identifications:

$$L^q = \dot{F}_q^{0,2}, \quad 1 < q < \infty, \tag{71}$$

and, more generally,

$$\dot{H}_q^s = \dot{F}_q^{s,2}, \quad s \in \mathbb{R}, 1 < q < \infty, \tag{72}$$

$$\dot{\mathcal{C}}^s = \dot{B}_\infty^{s,\infty}, \quad s \in \mathbb{R}, \tag{73}$$

$$\dot{F}_q^{0,2} = \mathcal{H}^q, \quad 0 < q \leqslant 1, \tag{74}$$

$$\dot{F}_\infty^{0,2} = BMO, \tag{75}$$

and

$$\dot{F}_\infty^{-1,2} = BMO^{-1}. \tag{76}$$

Moreover, we have the following continuous embedding (see [34]):

$$
3 \leqslant q_1 \leqslant q_2 < \infty,
$$
$$
L^3 \hookrightarrow \dot{B}_{q_1}^{-1+3/q_1,\infty} \hookrightarrow \dot{B}_{q_2}^{-1+3/q_2,\infty} \hookrightarrow \dot{F}_\infty^{-1,2} \hookrightarrow \dot{B}_\infty^{-1,\infty}. \tag{77}
$$

We will come back on the "maximal" space $\dot{B}_\infty^{-1,\infty}$ in Proposition 7.

The next four propositions are of paramount importance because they give definitions for the Besov and Triebel–Lizorkin norms in terms of the heat semigroup $S(t)$ (that appears in (12)) and in terms of the function Θ (that appears in (34) and (36)). The first two equivalences given hereafter, are very natural. The idea is that the convolution operators Δ_j can be interpreted as a discrete subset ($j \in \mathbb{Z}$) of the continuous set ($t > 0$) of convolution operators Θ_t where

$$\Theta_t = \frac{1}{t^3} \Theta \left(\frac{\cdot}{t} \right) \tag{78}$$

and, as in (35), Θ is defined by its Fourier transform $\widehat{\Theta}(\xi) = |\xi| e^{-|\xi|^2}$. If the function Θ were smooth and compactly supported on the Fourier side, this would indeed be the usual characterization for Besov and Triebel–Lizorkin spaces without any restriction on the third (regularity) index s that appears in Definitions 1 and 2. This would also be the case if the function Θ had all its moments equal to zero [185]. In the case we are dealing with, we only know that Θ has its first moment (the integral) equal to zero. This is why we have to require $s < 1$ (see [185]). The reader can consult [185] for the detailed proofs and [82,225,226] for a more general characterization.

PROPOSITION 1. *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s < 1$, *then the quantities*

$$\left(\sum_{j \in \mathbb{Z}} \left(2^{sj} \| \Delta_j f \|_q \right)^p \right)^{1/p} \tag{79}$$

and

$$\left(\int_0^\infty \left(t^{-s} \| \Theta_t f \|_q \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \tag{80}$$

are equivalent and will be referred to in the sequel by $\| f \|_{\dot{B}_q^{s,p}}$.

PROPOSITION 2. *Let* $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$ *and* $s < 1$, *then the quantities*

$$\left\| \left(\sum_{j \in \mathbb{Z}} (2^{sj} |\Delta_j f|)^p \right)^{1/p} \right\|_q \tag{81}$$

and

$$\left\| \left(\int_0^\infty (t^{-s} |\Theta_t f|)^p \frac{dt}{t} \right)^{1/p} \right\|_q \tag{82}$$

are equivalent and will be referred to in the sequel by $\| f \|_{\dot{F}_q^{s,p}}$.

The next two equivalences are even more useful because they allow us to pass from Δ_j to S_j (and from the discrete set S_j to the continuous $S(t)$ one). Here a restriction in the range of exponents also appears and we will be forced to assume that $s < 0$. More precisely, the reason why the equivalences under consideration are not true if $s \geqslant 0$ is essentially the following: even if we can easily estimate any quantity involving Δ_j from above with one only involving S_j, because of the identity

$$\Delta_j = S_{j+1} - S_j, \tag{83}$$

passing from Δ_j to S_j, via the relation

$$S_{j+1} = \sum_{k \leqslant j} \Delta_k, \tag{84}$$

it is not possible when $s \geqslant 0$ (see [185]). In the context of the Navier–Stokes equations, an explicit counter-example for $s = 0$ was given in [34] for the Besov spaces. A second one for the Triebel–Lizorkin spaces (always with $s = 0$) will be given in the following pages.

But let us state the equivalences we are talking about (for a proof see [225], p. 192).

PROPOSITION 3. *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s < 0$, *then the quantities*

$$\left(\sum_{j \in \mathbb{Z}} (2^{sj} \|\Delta_j f\|_q)^p \right)^{1/p}, \tag{85}$$

$$\left(\sum_{j \in \mathbb{Z}} (2^{sj} \|S_j f\|_q)^p \right)^{1/p}, \tag{86}$$

$$\left(\int_0^\infty (t^{-s/2} \|S(t) f\|_q)^p \frac{dt}{t} \right)^{1/p} \tag{87}$$

and

$$\left(\int_0^\infty \left(t^{-s} \| \Theta_t f \|_q \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \tag{88}$$

are equivalent and will be referred to in the sequel by $\| f \|_{\dot{B}_q^{s,p}}$.

PROPOSITION 4. *Let* $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$ *and* $s < 0$, *then the quantities*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(2^{sj} | \Delta_j f | \right)^p \right)^{1/p} \right\|_q, \tag{89}$$

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(2^{sj} | S_j f | \right)^p \right)^{1/p} \right\|_q, \tag{90}$$

$$\left\| \left(\int_0^\infty \left(t^{-s/2} | S(t) f | \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \right\|_q \tag{91}$$

and

$$\left\| \left(\int_0^\infty \left(t^{-s} | \Theta_t f | \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \right\|_q \tag{92}$$

are equivalent and will be referred to in the sequel by $\| f \|_{\dot{F}_q^{s,p}}$.

The next propositions will be also useful in the following pages. Of course the embeddings are also valid for inhomogeneous spaces.

PROPOSITION 5.

$$\text{If } s_1 > s_2 \text{ and } s_1 - \frac{3}{q_1} = s_2 - \frac{3}{q_2}, \text{ then } \dot{B}_{q_1}^{s_1, p_1} \hookrightarrow \dot{B}_{q_2}^{s_2, p_2} \text{ and } \dot{F}_{q_1}^{s_1, p_1} \hookrightarrow \dot{F}_{q_2}^{s_2, p_2}. \tag{93}$$

$$\text{If } p_1 < p_2, \text{ then } \dot{B}_q^{s_1, p_1} \hookrightarrow \dot{B}_q^{s_2, p_2} \text{ and } \dot{F}_q^{s_1, p_1} \hookrightarrow \dot{F}_q^{s_2, p_2}. \tag{94}$$

$$\text{For any } p, q \text{ and } s, \dot{B}_p^{s, \min(p,q)} \hookrightarrow \dot{F}_q^{s, p} \hookrightarrow \dot{F}_q^{s, \max(p,q)}. \tag{95}$$

2.3. *The paraproduct rule*

In order to study how the product acts on Besov spaces, we need to recall Bony's paraproduct algorithm [20], one of the most celebrated tools of paradifferential calculus. The Greek prefix "para" is added here in front of *product* and *differential* to underline that the new operations "go beyond" the usual ones. In particular, the new calculus enables us to define a new product between distributions which turns out to be continuous in many functional spaces where the usual product does not even make sense.

More precisely, let us consider two tempered distributions f and g and write, in terms of a Littlewood–Paley decomposition,

$$f = \sum_j \Delta_j f, \tag{96}$$

$$g = \sum_j \Delta_j g \tag{97}$$

so that, formally,

$$fg = \sum_n [S_{n+1} f S_{n+1} g - S_n f S_n g] + S_0 f S_0 g. \tag{98}$$

Now, after some simplifications, we get

$$fg = \sum_n [\Delta_n f S_n g + \Delta_n g S_n f + \Delta_n f \Delta_n g]$$

$$= \sum_n \Delta_n f S_{n-2} g + \sum_n \Delta_n g S_{n-2} f + \sum_{|n-n'| \leqslant 2} \Delta_{n'} f \Delta_n g. \tag{99}$$

In other words, the product of two tempered distributions is decomposed into two para-products, respectively,

$$\pi(f, g) = \sum_n \Delta_n f S_{n-2} g \tag{100}$$

and

$$\pi(g, f) = \sum_n \Delta_n g S_{n-2} f, \tag{101}$$

plus a remainder. Finally, if we want to analyze the product fg by means of the frequency filter Δ_j we deduce from (101), modulo some nondiagonal terms that we are neglecting for simplicity,

$$\Delta_j(fg) = \Delta_j f S_{j-2} g + \Delta_j g S_{j-2} f + \Delta_j \left(\sum_{k \geqslant j} \Delta_k f \Delta_k g \right). \tag{102}$$

Usually, the first two contributions are easier to treat than the third remainder term.

2.4. *The wavelet decomposition*

The Littlewood–Paley decomposition allows us to describe an arbitrary tempered distribution into the sum of regular functions that are well localized in the frequency variable.

The wavelet decomposition allows us to obtain an even better localization for these functions, say in both space and frequency. Of course, the ideal case of functions that are compactly supported in space as well as in frequency is excluded by Heisenberg's principle. Wavelets were discovered at the beginning of the 1980s and the best reference is Meyer's work [162,163].

The idea of using a wavelet decomposition to study turbulence questions was advocated from the very beginning, at about the same time when wavelets tools were available. In fact, due to the strong impact that wavelets had in several important scientific and technological discoveries, many people started dreaming that wavelets could provide the "golden rule" to attack the Navier–Stokes equations, from both mathematical and numerical points of view (see for instance the paper of Farge [77] and the references therein).

We do not discuss here the relevance of wavelets in numerical simulations of the Navier–Stokes equations and refer the reader to Meyer's conclusion in [166]. From the point of view of nonlinear partial differential equations, the situation is a little disappointing. The first attempt to approach the Navier–Stokes equations, by expanding the unknown velocity field $v(t, x)$ into a wavelet basis in space variable, came from Federbush, who wrote an intriguing paper in 1993 [78]. The techniques and insights employed arose from the theory of phase cell analysis used in constructive quantum field theory, and were the starting point and the first source of inspiration of our work [34].

The disappointing note is that, as we will see in the following sections, Federbush's program can be realized as well by using the less sophisticated Littlewood–Paley decomposition. On the other hand, the good news is that the systematic use of harmonic analysis tools (Littlewood–Paley and wavelets decomposition and their natural companions, Besov spaces and Bony's paraproducts techniques) paved the way for important discoveries for Navier–Stokes: the existence of a global solution for highly oscillating data, the uniqueness of this solution and its asymptotic behavior, via the existence of self-similar solutions.

As we have already announced in the Introduction, our story is full of surprises and bad news follows here at once. In fact, each proof of the previous results originally discovered by means of 'Fourier analysis methods', more precisely, by using 'Besov spaces', was followed shortly after its publication by a 'real variable methods' proof.

We will come back to these questions – existence, uniqueness, self-similar solutions – and treat them in detail in three separate sections (resp. Sections 4–6). Before doing this and in order to clarify the previous discussion, let us briefly recall here, for the convenience of the reader, some definitions taken from the wavelet world. Roughly speaking, a wavelet decomposition is a decomposition of the type

$$f = \sum_\lambda \langle f, \psi_\lambda \rangle \psi_\lambda, \tag{103}$$

where ψ_λ is "essentially" localized in frequency in a dyadic annulus 2^j and "essentially" localized in space in a dyadic cube 2^{-j}. More precisely, following Meyer [162], we have the following definition:

DEFINITION 8. A wavelet decomposition of regularity $m > 0$ is a set of $2^3 - 1 = 7$ functions ψ_ε, $\varepsilon \in \{0, 1\}^3 \backslash \{0, 0, 0\}$ verifying the following properties:

1. *Regularity*: ψ_ε belongs to $\mathcal{C}^m$.
2. *Localization*:

$$\forall \alpha, |\alpha| \leqslant m, \forall N \in \mathbb{N}, \exists C: \quad \left|\partial^\alpha \psi_\varepsilon\right|(x) \leqslant C\left(1 + |x|\right)^{-N}. \tag{104}$$

3. *Oscillation*:

$$\forall \alpha, |\alpha| \leqslant m: \quad \int x^\alpha \psi_\varepsilon(x)\, \mathrm{d}x = 0. \tag{105}$$

4. *Orthogonality*: The set

$$\left\{2^{3j/2}\psi_\varepsilon\left(2^j x - k\right) \middle/ j \in \mathbb{Z}^3, \varepsilon \in \{0,1\}^3 \backslash \{0,0,0\}\right\} \tag{106}$$

is an orthogonal basis of L^2.

If we denote $\psi_{j,k}(x) = 2^{3j/2}\psi(2^j x - k)$ (where, for the sake of simplicity, the parameter ε is neglected), then we obtain the following "homogeneous" decomposition

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3} \langle f, \psi_{j,k} \rangle \psi_{j,k} = \sum_{j,k} c_{j,k} \psi_{j,k} \tag{107}$$

that, as in the case of the homogeneous Littlewood–Paley decomposition, has to be understood in $\mathcal{S}'$ modulo polynomials.

Formally, a Littlewood–Paley decomposition Δ_j gives a wavelet decomposition $\psi_{j,k}$ by letting

$$c_{j,k} \simeq \Delta_j f\left(2^{-j}k\right) \tag{108}$$

and, vice versa, from a wavelet decomposition we can recover a Littlewood–Paley one just by taking

$$\Delta_j f \simeq \sum_{k \in \mathbb{Z}^3} c_{j,k} \psi_{j,k}. \tag{109}$$

Finally, the wavelets coefficients $c_{j,k}$ of a function f allow us to obtain an equivalent definition of the Besov and Triebel–Lizorkin spaces. For example, we have the following proposition [162]:

PROPOSITION 6. *If ψ is a wavelet of regularity $m > 0$, then, for any $|s| < m$ and any $1 \leqslant p, q \leqslant \infty$, we have the equivalence of norms*

$$\|f\|_{\dot{B}_q^{s,p}} \simeq \left(\sum_{j \in \mathbb{Z}} 2^{jp(s+3(1/2-1/q))} \left(\sum_{k \in \mathbb{Z}^3} |c_{j,k}|^q\right)^{p/q}\right)^{1/p}. \tag{110}$$

REMARK. In the study of the Navier–Stokes equations and other *incompressible* fluid equations, one would expect that the wavelets functions ψ_ε in Definition 8 have an additional property:

5. *Divergence-free*: Divergence-free basis of wavelets were first discovered by Battle and Federbush [6,7] and their construction was improved later by Lemarié [139,140,145]. A simple presentation of these basis is contained in the paper by Meyer [166].

2.5. *Other useful function spaces*

Before we enter the heart of the paper, devoted to existence and uniqueness theorems for the Navier–Stokes equations, we wish to end this section by presenting other functional spaces, that will be useful in the following pages.

2.5.1. *Morrey–Campanato spaces.* For $1 \leqslant q \leqslant p \leqslant \infty$, the inhomogeneous Morrey–Campanato space M_q^p is defined as the space of functions f which are locally in L^q and such that

$$\sup_{x \in \mathbb{R}^3, 0 < r \leqslant 1} R^{3/p} \left(R^{-3} \int_{|x-y| \leqslant r} \left| f(x) \right|^q \, \mathrm{d}y \right)^{1/q} < \infty, \tag{111}$$

where the left-hand side of this inequality is the norm of f in M_q^p. The homogeneous Morrey–Campanato space $\dot{M}_q^p$ is defined in the same way, by taking the supremum over all $r \in (0, \infty)$ instead of $r \in (0, 1]$.

2.5.2. *Lorentz spaces.* Let $1 \leqslant p, q \leqslant \infty$, then a function f belongs to the Lorentz space $L^{(p,q)}$ if and only if 'the quantity'

$$\|f\|_{L^{(p,q)}} = \left(\frac{q}{p} \int_0^\infty \left[t^{1/p} f^*(t) \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty, \tag{112}$$

of course, if $q = \infty$ this means

$$\|f\|_{L^{(p,\infty)}} = \sup_{t>0} t^{1/p} f^*(t) < \infty, \tag{113}$$

where f^* is the decreasing rearrangement of f:

$$f^*(t) = \inf \left\{ s \geqslant 0; \left| \left\{ |f| > s \right\} \right| \leqslant t \right\}, \quad t \geqslant 0. \tag{114}$$

We know [215] that for $p > 1$, a norm on $L^{(p,q)}$ equivalent to 'the quantity' $\|f\|_{L^{(p,q)}}$ exists such that $L^{(p,q)}$ becomes a Banach space. If $p = q$, the space $L^{(p,p)}$ is nothing else than the Lebesgue space L^p. Moreover, generalization versions of Hölder and Young

inequalities hold for the Morrey–Campanato spaces [111]. Finally, for these spaces, the theory of real interpolation gives the equivalence (see [10])

$$\left(L^{p_0}, L^{p_1}\right)_{(\theta,q)} = L^{(p,q)}, \tag{115}$$

where $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ satisfy $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1 \leqslant q \leqslant \infty$.

2.5.3. *Le Jan–Sznitman spaces.*   Recently, Le Jan and Sznitman [137,138] considered the space of tempered distributions f whose Fourier transform verifies

$$\sup_{\mathbb{R}^3} |\xi|^2 \left|\hat{f}(\xi)\right| < \infty. \tag{116}$$

Now, if in the previous expression we consider $\int_{\xi \in \mathbb{R}^3}$ instead of $\sup_{\xi \in \mathbb{R}^3}$, we obtain the (semi)-norm of a homogeneous Sobolev space. This is not the case: the functions whose Fourier transform is bounded define the pseudo-measure space $\mathcal{PM}$ of Kahane. In other words, a function f belongs to the space introduced by Le Jan and Sznitman if and only if $\Delta f \in \mathcal{PM}$, Δ being the Laplacian (in three dimensions). A simple calculation (see [48]) shows that condition (116) is written, in the dyadic decomposition Δ_j of Littlewood and Paley in the form $4^j \|\Delta_j f\|_{\mathcal{PM}} = 4^j \|\widehat{\Delta_j f}\|_\infty \in \ell^\infty(\mathbb{Z})$ and defines in this way "the homogeneous Besov space" $\dot{B}^{2,\infty}_{\mathcal{PM}}$.

Let us note that this quantity is not a norm, unless we work in $\mathcal{S}'$ modulo polynomials, as we did in Section 2.2 in the case of homogeneous Besov spaces (for example, if f is a constant or, more generally a polynomial of degree 1, it is easy to see that $|\xi|^2 |\hat{f}(\xi)| = 0$). Another possibility to avoid this technical point is to ask that $\hat{f} \in L^1_{\text{loc}}$. In other words, the Banach functional space relevant to our study is defined by

$$\mathcal{PM}^2 = \left\{ v \in \mathcal{S}' : \hat{v} \in L^1_{\text{loc}}, \|v\|_{\mathcal{PM}^2} \equiv \sup_{\xi \in \mathbb{R}^3} |\xi|^2 \left|\hat{v}(\xi)\right| < \infty \right\}. \tag{117}$$

A generalization of this functional space was recently introduced in the paper by Bhattacharya, Chen, Dobson, Guenther, Orum, Ossiander, Thomann and Waymire (see [8]).

3. Existence theorems

3.1. *The fixed point theorem*

We will recall here two classical results concerning the existence of fixed point solution to abstract functional equations. These theorems are known under the name of Picard in France, Caccioppoli in Italy, and Banach in Poland and . . . in the rest of the world!

LEMMA 4. *Let X be an abstract Banach space with norm $\| \cdot \|$ and $B : X \times X \to X$ a bilinear operator such that, for any $x_1, x_2 \in X$,*

$$\big\| B(x_1, x_2) \big\| \leqslant \eta \|x_1\| \|x_2\|, \tag{118}$$

then, for any $y \in X$ such that

$$4\eta \|y\| < 1, \tag{119}$$

the equation

$$x = y + B(x, x) \tag{120}$$

has a solution x in X. In particular, the solution is such that

$$\|x\| \leqslant 2\|y\| \tag{121}$$

and it is the only one such that

$$\|x\| < \frac{1}{2\eta}. \tag{122}$$

The following lemma is a generalization of the previous one ($\lambda = 0$) and will be useful when treating the mild Navier–Stokes equations in the presence of a nontrivial external force (11).

LEMMA 5. *Let X be an abstract Banach space with norm $\| \cdot \|$, $L : X \to X$ a linear operator such that, for any $x \in X$,*

$$\big\| L(x) \big\| \leqslant \lambda \|x\| \tag{123}$$

and $B : X \times X \to X$ a bilinear operator such that, for any $x_1, x_2 \in X$,

$$\big\| B(x_1, x_2) \big\| \leqslant \eta \|x_1\| \|x_2\|, \tag{124}$$

then, for any λ, $0 < \lambda < 1$, and for any $y \in X$ such that

$$4\eta \|y\| < (1 - \lambda)^2, \tag{125}$$

the equation

$$x = y + B(x, x) + L(x) \tag{126}$$

has a solution x in X. In particular, the solution is such that

$$\|x\| \leqslant \frac{2\|y\|}{1 - \lambda} \tag{127}$$

and it is the only one such that

$$\|x\| < \frac{1-\lambda}{2\eta}. \tag{128}$$

For an elementary proof of the above mentioned lemmata the reader is referred to [34] and to [3] where a different proof is given that also applies to the (optimal) case where the equality sign holds in (119), (122), (125) and (128).

3.2. *Scaling invariance*

The Navier–Stokes equations are invariant under a particular change of time and space scaling. More exactly, assume that, in $\mathbb{R}^3 \times (0, \infty)$, $v(t, x)$ and $p(t, x)$ solve the system

$$\frac{\partial v}{\partial t} - \nu \Delta v = -(v \cdot \nabla)v - \nabla p,$$
$$\nabla \cdot v = 0, \tag{129}$$

then the same is true for the rescaled functions

$$v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x), \qquad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x). \tag{130}$$

On the other hand, the functions $v(\lambda t, \lambda x)$ and $p(\lambda t, \lambda x)$ solve a different Navier–Stokes system, where ν is replaced by $\lambda \nu$, thus allowing us to assume that viscosity is equal to unity, as we did in Section 1.1 (because, if not, it is possible to find a $\lambda > 0$ such that $\lambda \nu = 1$). The above scaling invariance leads to the following definition.

DEFINITION 9. *Critical space*. A translation invariant Banach space of tempered distributions X is called a critical space for the Navier–Stokes equations if its norm is invariant under the action of the scaling $f(x) \to \lambda f(\lambda x)$ for any $\lambda > 0$. In other words, we require the embedding

$$X \hookrightarrow \mathcal{S}' \tag{131}$$

and that, for any $f \in X$,

$$\|f(\cdot)\| = \|\lambda f(\lambda \cdot -x_0)\| \quad \forall \lambda > 0, \forall x_0 \in \mathbb{R}^3. \tag{132}$$

Critical spaces are all embedded in a same function space, as stated in the following proposition.

PROPOSITION 7 (A remarkable embedding). *If X is a critical space, then X is continuously embedded in the Besov space $\dot{B}_\infty^{-1,\infty}$.*

The proof of this result is so simple that we would like to present it here. We argue as in the proof of the "minimality of $\dot{B}_1^{0,1}$" given by Frazier, Jawerth and Weiss in [82] (see also [3,161,166]).

To begin, we note that if X satisfies (131), then there exists a constant C such that

$$\left| \left\langle \exp\left(-|x|^2/4\right), f \right\rangle \right| \leqslant C \|f\|_X \quad \forall f \in X. \tag{133}$$

Now, using the translation invariance of X we obtain

$$\left\| \exp(\Delta) f \right\|_{L^\infty} \leqslant C \|f\|_X \tag{134}$$

and, by the invariance under the scaling $f(x) \to \lambda f(\lambda x)$, we get

$$t^{1/2} \left\| \exp(t\Delta) f \right\|_{L^\infty} \leqslant C \|f\|_X. \tag{135}$$

It is now easy to conclude if we recall Proposition 3, say

$$\sup_{t>0} t^{1/2} \left\| \exp(t\Delta) f \right\|_{L^\infty} \simeq \|f\|_{\dot{B}_\infty^{-1,\infty}}. \tag{136}$$

As we will see in the following pages, it is a remarkable feature that the Navier–Stokes equations are well posed in the sense of Hadamard (existence, uniqueness and stability) when the initial data is divergence-free and belongs to certain *critical* function spaces. Actually, it is unclear whether this property is true for either a generic critical space or for the bigger critical space $\dot{B}_\infty^{-1,\infty}$ (see the conjecture formulated in [166], Chapter 8, and [160]), but it happens to be the case for most of the critical functional spaces we have described so far.

For example, in the Lebesgue family $L^p = L^p(\mathbb{R}^3)$ the critical invariant space corresponds to the value $p = 3$ (more generally in $\mathbb{R}^n$, $p = n$) and we will see how to construct mild solutions to the Navier–Stokes equations with data in L^3. The same argument applies to the critical Sobolev space $\dot{H}^{1/2}$, to the Morrey–Campanato $\dot{M}_p^3$ ($1 \leqslant p \leqslant 3$), the Lorentz $L^{(3,q)}$ ($1 \leqslant q \leqslant \infty$), the pseudo-measure space of Le Jan and Sznitman $\mathcal{PM}^2$, the Besov $\dot{B}_p^{3/p-1,q}$ ($1 \leqslant q \leqslant \infty$, $1 \leqslant p < \infty$) as well as the Triebel–Lizorkin spaces $\dot{F}_p^{3/p-1,q}$ ($1 \leqslant q < \infty$, $1 \leqslant p < \infty$). The reader is referred to [3] for a precise and exhaustive analysis of the Navier–Stokes equations in critical spaces. Here we will only treat the case of the Lebesgue space L^3 in detail.

Another (equivalent) way of defining critical spaces for the Navier–Stokes equations is to note that in this case the nonlinear term $\nabla \cdot (v \otimes v)$ has the same strength as the Laplace operator; that is $\nabla \cdot (v \otimes v)$ is not subordinate to $-\Delta v$. For instance, if $v \in L^p$ ($p \geqslant 2$), then $\nabla \cdot (v \otimes v) \in W^{p/2,-1}$ whereas $-\Delta v \in W^{p,-2}$ and, by Sobolev embedding, $W^{p/2,-1} \hookrightarrow W^{p,-2}$ as long as $p \geqslant 3$.

Before recalling the main steps of the proof for the existence of mild solution with initial data in L^3, let us begin with an easier case, the so-called 'supercritical' space L^p, $p > 3$. We will not give a precise definition of 'critical', 'supercritical', or 'subcritical' spaces. The meaning of their names should be clear enough to any reader (for more details and

examples see [34,43]). Let us just notice that what we call here 'supercritical' spaces are called 'subcritical' spaces (and vice versa) in the paper by Klainerman [122].

3.3. *Supercritical case*

The main theorem of the existence of mild solutions in L^p, $3 < p < \infty$, was known since the papers of Fabes, Jones and Rivière [76] (1972) and Giga [100] (1986). Concerning the space L^∞, let us note that the existence was obtained only recently in [34,43] by using the simplified structure of the bilinear term we introduced in (34). In fact, as pointed out in a different proof by Giga and his students in Sapporo [104], the difficulty comes from the fact that the Leray–Hopf projection $\mathbb{P}$ is not bounded in L^∞, nor in L^1. The proof we are going to present applies to $3 < p \leqslant \infty$ and is contained in [34,43]. The idea is of course to use the fixed point theorem by means of the following two lemmata, whose proofs are obtained by a simple application of the Young inequality.

LEMMA 6. *Let X be a Banach space, whose norm is translation invariant. For any $T > 0$ and any $v_0 \in X$, we have*

$$\sup_{0 < t < T} \| S(t)v_0 \|_X = \|v_0\|_X. \tag{137}$$

Of course this lemma applies for example when X is a Lebesgue space, in our case $X = L^p$ with $3 < p \leqslant \infty$.

LEMMA 7. *Let $3 < p \leqslant \infty$ be fixed. For any $T > 0$ and any functions $f(t), g(t) \in \mathcal{C}([0, T]; L^p)$, then the bilinear term $B(f, g)(t)$ also belongs to $\in \mathcal{C}([0, T]; L^p)$ and we have*

$$\sup_{0 < t < T} \| B(f, g)(t) \|_p \lesssim \frac{T^{1/2(1-3/p)}}{1 - 3/p} \sup_{0 < t < T} \| f(t) \|_p \sup_{0 < t < T} \| g(t) \|_p. \tag{138}$$

Combining these lemmata with the fixed point algorithm Lemma 4 we obtain the following existence result (see Section 5.2 for its uniqueness counterpart).

THEOREM 1. *Let $3 < p \leqslant \infty$ be fixed. For any $v_0 \in L^p$, $\nabla \cdot v_0 = 0$, there exists a $T = T(\|v_0\|_p)$ such that the Navier–Stokes equations has a solution in $\mathcal{C}([0, T]; L^p)$.*

To be more precise, according to the notations introduced in Definition 2, we should write $v \in \mathcal{C}([0, T]; \mathbb{P}L^p)$, because the solution constructed so far is of course a solenoidal (i.e., divergence-free) vector field. To simplify the discussion, we prefer not to use such notation in the following.

We should also remark that the strong continuity at $T = 0$ is not ensured in the case L^∞, because this space is nonseparable. In other words, if it is true that

$$\lim_{t \to 0} \| v(t) - v_0 \|_p = 0, \quad 3 < p < \infty, \tag{139}$$

this is not the case if $p = \infty$, for the heat semigroup $S(t)$ is not strongly continuous as $t \to 0$.

There are two ways to restore continuity in the case of a nonseparable Banach space X. The first is to restrict the attention to X_*, the closure of C_0^∞ in X. Then, $S(t)$ is strongly continuous and the existence theorem applies as stated. On the other hand, if X is nonseparable, but instead X is the dual of a separable space Y (here $X = L^\infty$ and $Y = L^1$), it is natural to replace $\mathcal{C}([0, T); X)$ with the space we will denote $\mathcal{C}_*([0, T); X)$ consisting of *bounded* functions $v(t)$ with values in X which have the property that v is continuous in t with values in X, when X is endowed with the $\sigma(X, Y)$ topology (see [34,43,104,166,216]).

Finally, we will see in the next section that the solution constructed so far is always regular, unique and stable. This means that the Cauchy problem is *locally in time* well posed if the data belong to the supercritical space L^p, $3 < p \leqslant \infty$. It is an open question to know whether the solution is actually global in time. The noninvariance of the L^p norm, $p \neq 3$ ensures that such a global result would not depend on the size of the initial data, say the quantity $\|v_0\|_p$ (or, more generally, if $\nu \neq 1$, the quantity $\|v_0\|_p/\nu$).

3.4. *Critical case*

By means of the critical Lebesgue space L^3 we will see how to construct the existence not only of local solutions for arbitrary initial data, but also of global ones, for small or highly oscillating data (this property will be described in detail in Section 4).

Let us begin with an unpleasant remark. If we try to apply the fixed point theorem to the integral Navier–Stokes equation

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (v \otimes v)(s)\,\mathrm{d}s \tag{140}$$

in the (natural) function space

$$\mathcal{N} = \mathcal{C}\big([0, T); L^3\big), \tag{141}$$

we are faced with a difficulty that did not appear in the supercritical case: the bilinear term $B(v, u) = -\int_0^t S(t-s)\mathbb{P}\nabla \cdot (v \otimes u)(s)\,\mathrm{d}s$ is not continuous from $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$.

Of course, the fact that the estimate (138) diverges when $p = 3$ is not enough to show the noncontinuity: first, we would expect a reverse inequality, second, this reverse inequality should apply to the full vectorial bilinear term (in fact, in a way reminiscent of the so-called "div–curl" lemma [66], one can imagine that the full bilinear operator is continuous even if its simplified scalar version is not).

In his unpublished doctoral thesis [183], Oru proved the noncontinuity of the full vectorial term not only in the Lebesgue space L^3, but also in any Lorentz space $L^{(3,q)}$, for any $q \in [1, \infty)$:

PROPOSITION 8. *The* (*vectorial*) *bilinear operator B is not continuous from $\mathcal{C}([0, T);$ $L^{(3,q)}) \times \mathcal{C}([0, T); L^{(3,q)}) \to \mathcal{C}([0, T); L^{(3,q)})$, whatever $0 < T \leqslant \infty$ and $q \in [1, \infty)$ are.*

At about the same time Meyer [166] showed that the critical space $L^{(3,\infty)}$ is very different since:

PROPOSITION 9. *The bilinear operator B is continuous from* $\mathcal{C}([0,T); L^{(3,\infty)}) \times \mathcal{C}([0,T); L^{(3,\infty)}) \to \mathcal{C}([0,T); L^{(3,\infty)})$ *for any* $0 < T \leqslant \infty$.

Oru's theorem is based on the following remark (see also [145]):

LEMMA 8. *If X is a critical space in the sense of Definition* 9 *and if the bilinear operator B is continuous in the space* $\mathcal{C}([0,T); X)$ *for a certain T, then X contains a function of the form*

$$\frac{\omega(x)}{|x|} + \phi(x), \tag{142}$$

where ω does not vanish identically, is homogeneous of degree 0, *is C^∞ outside the origin and ϕ is a C^∞ function.*

In fact, it is possible to prove that functions of the type (142) do not belong to $L^{(3,q)}$, if $q \neq \infty$ but can be in $L^{(3,\infty)}$, thus not contradicting Proposition 9.

Let us note, in passing, that it is very surprising that for a generic critical space we cannot be sure whether the bilinear term is continuous or not. Another example where it is quite easy to prove the continuity of the bilinear term (and thus the existence of a solution) is provided by the critical space $\mathcal{PM}^2$ introduced by Le Jan and Sznitman [138]. We will describe some important consequences of the continuity of the bilinear term in the spaces $L^{(3,\infty)}$ and $\mathcal{PM}^2$ in Sections 6.2 and 6.4.

Let us go back to L^3. If we want to find a mild solution with initial data in this space, there are (at least) three ways to circumvent the obstacle arising from Proposition 8 and are all based on the following remark: the fixed point algorithm in $\mathcal{N}$ is only a *sufficient* condition to ensure the existence of a solution in $\mathcal{N}$ and a different strategy can be considered.

To be more explicit, another sufficient condition leading to the existence of a solution in $\mathcal{N}$ is to find a function space $\mathcal{F}$ (whose elements are functions $v(t,x)$ with $0 < t < T$ and $x \in \mathbb{R}^3$) such that:

(1) the bilinear term $B(u,v)(t)$ is continuous from $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$,
(2) if $v_0 \in L^3$, then $S(t)v_0 \in \mathcal{F}$, and
(3) the bilinear term $B(u,v)(t)$ is continuous from $\mathcal{F} \times \mathcal{F} \to \mathcal{N}$.

In fact, the first two conditions ensure the existence of a (mild) solution $v(t,x) \in \mathcal{F}$, via the fixed point algorithm and, thanks to the third condition, this solution belongs to $\mathcal{N}$ as well (if $\mathcal{F} \hookrightarrow \mathcal{N}$, the third condition being of course redundant).

The three ways known in the literature to obtain a solution $v(t,x) \in \mathcal{N}$ with data in L^3 correspond to three different choices of spaces $\mathcal{F}$ [47]. For the convenience of the reader we will briefly recall in the following sections these spaces leading to the same existence theorem in $\mathcal{N}$ that reads as follows.

THEOREM 2. *For any $v_0 \in L^3$, $\nabla \cdot v_0 = 0$, there exists a $T = T(v_0)$ such that the Navier–Stokes equations have a local solution in $\mathcal{C}([0, T); L^3)$. Moreover, there exists $\delta > 0$ such that if $\|v_0\|_3 < \delta$, then the solution is global, i.e., we can take $T = \infty$.*

As it will be clear in the following pages, here at variance with Theorem 1 we cannot say that $T = T(\|v_0\|_3)$. Again, as far as the uniqueness of the solution, the situation is more delicate and will be revealed in Section 5.3.

3.4.1. *Weissler's space.* In 1981, Weissler [234] gave the first existence result of mild solutions in the half space $L^3(\mathbb{R}^3_+)$, then Giga and Miyakawa [106] generalized the proof to $L^3(\Omega_b)$, Ω_b an open bounded domain in $\mathbb{R}^3$. Finally, in 1984, Kato [114] obtained, by means of a purely analytical proof (involving only Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$.

In [34,35,47] we showed how to simplify Kato's proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form, as in (34) and (36). In particular, the divergence $\nabla\cdot$ and heat $S(t)$ operators can be treated as a single convolution operator [34]. This is why no explicit conditions on the gradient of the unknown function v and no restriction on q (namely $3 < q < 6$) will be required here, as they were indeed in Kato's original paper [114]. In a different context [34,43] and by using the same simplified scalar structure, it was possible to show the existence of a solution with data in the Lebesgue space L^∞ (Section 3.3), even if the Leray–Hopf operator $\mathbb{P}$ is not bounded in L^∞.

In order to proceed, we have to recall the definition of the auxiliary space $\mathcal{K}_q$ ($3 \leqslant q \leqslant \infty$) introduced by Weissler and systematically used by Kato. More exactly, this space $\mathcal{K}_q$ is made up by the functions $v(t, x)$ such that

$$t^{\alpha/2} v(t, x) \in \mathcal{C}\big([0, T); L^q\big) \tag{143}$$

and

$$\lim_{t \to 0} t^{\alpha/2} \big\|v(t)\big\|_q = 0, \tag{144}$$

with q being fixed in $3 < q \leqslant \infty$ and $\alpha = \alpha(q) = 1 - 3/q$. In the case $q = 3$, it is also convenient to define the space $\mathcal{K}_3$ as the natural space $\mathcal{N}$ with the additional condition that its elements $v(t, x)$ satisfy

$$\lim_{t \to 0} \big\|v(t)\big\|_3 = 0. \tag{145}$$

The theorem in question, that implies Theorem 2, is the following [34]:

THEOREM 3. *Let $3 < q < \infty$, and $\alpha = 1 - 3/q$ be fixed. There exists a constant $\delta_q > 0$ such that, for any initial data $v_0 \in L^3$, $\nabla \cdot v_0 = 0$ in the sense of distributions such that*

$$\sup_{0 < t < T} t^{\alpha/2} \big\|S(t)v_0\big\|_q < \delta_q, \tag{146}$$

*then there exists a mild solution $v(t, x)$ to the Navier–Stokes equations belonging to $\mathcal{N}$,
which tends strongly to v_0 as time goes to zero. Moreover, this solution belongs to all
spaces $\mathcal{K}_q$ for all $3 < q < \infty$. In particular*, (146) *holds for arbitrary $v_0 \in L^3$ provided we
consider $T(v_0)$ small enough, and as well if $T = \infty$, provided the norm of v_0 in the Besov
space $\dot{B}_q^{-\alpha,\infty}$ is smaller than δ_q.*

The existence part of the proof of this theorem is a consequence of the following lem-
mata that we recall here.

LEMMA 9. *If $v_0 \in L^3$, then $S(t)v_0 \in \mathcal{K}_q$ for any $3 < q \leqslant \infty$. In particular this implies*
(*when $T = \infty$*) *the continuous embedding*

$$L^3 \hookrightarrow \dot{B}_q^{-\alpha,\infty}, \quad 3 < q \leqslant \infty. \tag{147}$$

In particular, this lemma implies that the conclusion of Theorem 3 holds not only in
the general case of arbitrary $v_0 \in L^3$ when $T = \infty$, provided the norm of v_0 in the Besov
space $\dot{B}_q^{-\alpha,\infty}$ is smaller than δ_q, but also in the more restrictive case of $v_0 \in L^3$ and small
enough in L^3, as we recalled in the statement of Theorem 2 and originally proved in the
papers of Weissler, Giga and Miyakawa, and Kato. In other words, a function in L^3 can
be arbitrarily large in the L^3 norm but small in $\dot{B}_q^{-\alpha,\infty}$. This remark will play a key role
in Section 4. Another important consequence of this lemma is that L^3 and $B_q^{-\alpha,\infty}$ are
different spaces, for $|x|^{-1} \in \dot{B}_q^{-\alpha,\infty}$ and $|x|^{-1} \notin L^3$ and this will allow the construction of
self-similar solutions in Section 6.

The second lemma we need in order to prove Theorem 3 is the following:

LEMMA 10. *The bilinear operator $B(f, g)(t)$ is bicontinuous from $\mathcal{K}_q \times \mathcal{K}_q \to \mathcal{K}_q$ for
any $3 < q < \infty$.*

Once these two lemmata are applied for a *certain q*, $3 < q < \infty$, one can easily deduce,
provided (146) is satisfied and via the fixed point algorithm, the existence of *a solution*
$v(t, x) \in \mathcal{N}$ that tends strongly to v_0 at zero and belongs to $\mathcal{K}_q$ for *all $3 < q < \infty$*.

The latter properties are a consequence of the following generalization of Lemma 10,
applied to the bilinear B term.

LEMMA 11. *The bilinear operator $B(f, g)(t)$ is bicontinuous from $\mathcal{K}_q \times \mathcal{K}_q \to \mathcal{K}_p$ for
any $3 \leqslant p < \frac{3q}{6-q}$ if $3 < q < 6$; any $3 \leqslant p < \infty$ if $q = 6$; and $q/2 \leqslant p \leqslant \infty$ if $6 < q < \infty$.*

The proof of the uniqueness of the solution in $\mathcal{N}$ requires a more careful study of the
bilinear term as it will be explained in Section 5.3.

Before moving on to a different strategy to prove Theorem 2, let us mention here that
the limit value $q = \infty$ cannot be considered in the statement of Lemma 10 because, if we
use the standard approach to prove the continuity in L^∞, we are led to a divergent integral

(see [166], Chapter 19). Thus, a priori, it is not possible to deduce the existence of a mild solution in $\mathcal{N}$ when the condition expressed by (146) is satisfied for $q = \infty$, say

$$\sup_{0<t<T} t^{1/2} \big\| S(t)v_0 \big\|_\infty < \delta, \tag{148}$$

(which means, when $T = \infty$, that the norm of v_0 in the Besov space $\dot{B}_\infty^{-1,\infty}$ is small enough). If, instead, we just require the strongest condition

$$\sup_{0<t<T} t^{1/2} \big\| S(t)v_0 \big\|_\infty + \sup_{0<t<T} \big\| S(t)v_0 \big\|_3 < \delta \tag{149}$$

(which means that, when $T = \infty$, the norm of v_0 in L^3 is small enough), then the existence of a mild solution $v(t,x)$ belonging to $\mathcal{N}$ can be ensured. Moreover, this solution belongs to $\mathcal{K}_\infty$.

Once again, it is obvious that this result implies Theorem 2, at least when $T = \infty$. At difference with the proof of Theorem 3, here we cannot apply the fixed point theorem directly in $\mathcal{K}_\infty$, but in the space $\mathcal{K}$ whose elements are functions $v(t,x)$ belonging to the intersection $\mathcal{K}_\infty \cap \mathcal{N}$ and whose norm is given by $\sup_{0<t<T} t^{1/2} \|v(t)\|_\infty + \sup_{0<t<T} \|v(t)\|_3$. In fact, the following lemma:

LEMMA 12. *The bilinear operator $B(f,g)(t)$ is bicontinuous from $\mathcal{K} \times \mathcal{K} \to \mathcal{K}$,*

whose proof is contained, for example, in [166], holds true and allows us to conclude.

3.4.2. *Calderón's space.*   Another way to prove the existence of a solution with data in L^3 was discovered by Calderón [32] in 1990 and was independently proposed five years later in [34] (see [37] for more details).

Here the auxiliary function space will be denoted by the letter $\mathcal{M}$. Its elements $v(t,x)$ are such that

$$\big\| |v| \big\|_{\mathcal{M}} = \Big\| \sup_{0<t<T} |v(t,x)| \Big\|_3 \tag{150}$$

is finite.

It is easy to see that $\mathcal{M}$ is continuously embedded in $\mathcal{N}$, because of the following elementary inequality

$$\sup_{0<t<T} \big\| v(t,x) \big\|_3 \leqslant \Big\| \sup_{0<t<T} |v(t,x)| \Big\|_3. \tag{151}$$

The method we will pursue here is to solve the mild Navier–Stokes equations in $\mathcal{M}$. This will be possible because, at variance with $\mathcal{N}$, the bilinear operator is bicontinuous in $\mathcal{M}$. More precisely, the following two lemmata hold true [32–34].

LEMMA 13. $S(t)v_0 \in \mathcal{M}$ *if and only if $v_0 \in L^3$.*

This lemma, whose proof follows from Hardy–Littlewood maximal function, shows that the equivalence stated in Proposition 4 is not true if for example $s = 0$, $p = \infty$ and $q = 3$. In fact, the equivalence under consideration can be seen as a consequence of the well-known result that the Hardy space $\mathcal{H}^3$ is equivalent to L^3, which in turn is equivalent to the Triebel–Lizorkin space $\dot{F}_3^{0,2}$. For a more detailed explanation on this subject we refer the reader to [225,226].

The following lemma concerns the bilinear term [32–34].

LEMMA 14. *The bilinear operator $B(f, g)(t)$ is bicontinuous from $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$.*

Before proceeding, we want to make an additional comment here. The fact that the bilinear operator $B(f, g)$ is bicontinuous both in $\mathcal{M}$ (that *is included* in $\mathcal{N}$) and, as it was announced by Meyer [166], bicontinuous in the Lorentz space $\mathcal{C}([0, T); L^{(3,\infty)})$ (that *includes* $\mathcal{N}$), is very peculiar, since Oru showed in [183] that $B(f, g)$ is not bicontinuous in the natural space $\mathcal{N}$.

This remark being made, let us see how, by a simple variant of the proof above, one can generalize Lemma 14. In order to do that, let us introduce the space $\mathcal{H}_p^s$ whose elements $v(t, x)$ are such that

$$\|v\|_{\mathcal{H}_p^s} = \left\| \sup_{0<t<T} \left| \dot{\Lambda}^s v(t, x) \right| \right\|_p < \infty. \tag{152}$$

Here $\dot{\Lambda}^s$ is as usual the pseudo-differential operator whose symbol in Fourier transform is given by $|\xi|^s$ and $\dot{\Lambda} = \sqrt{-\Delta}$ is the Calderón operator.

In other words, $\mathcal{H}_p^s$ is the subspace of the natural space $\mathcal{C}([0, T); \dot{H}_p^s)$ obtained by interchanging the time and space norms. Here, $\dot{H}_p^s = \dot{F}_p^{s,2}$ corresponds to the so-called Sobolev–Bessel or homogeneous Lebesgue space. In particular, for $p < 3$, we have the following continuous embedding,

$$\dot{H}_p^{3/p-1} \hookrightarrow L^3 = \dot{H}_3^0 \tag{153}$$

which, in turn, gives ($p < 3$)

$$\mathcal{H}_p^{3/p-1} \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{N}. \tag{154}$$

We are now ready to generalize Lemma 14 ($p = 3$) in the following:

LEMMA 15. *Let $3/2 < p < 3$ be fixed. The bilinear operator $B(f, g)(t)$ is bicontinuous from $\mathcal{M} \times \mathcal{M} \to \mathcal{H}_p^{3/p-1}$.*

This lemma should be interpreted as a supplementary regularity property of the bilinear term as it was extensively analyzed in [34,43,48,186]. By means of a more accurate study of the cancellation properties of the bilinear term, the limit case $p = 3/2$ (with the natural norm in time and space variables) can be included as well (see [48]).

This remark being made, let us observe that, just by using Lemmas 13 and 14, we are in a position, via the fixed point algorithm, to prove the existence of a global solution in $\mathcal{M}$ with initial data v_0 sufficiently small in L^3, say

$$\|v_0\|_3 < \delta. \tag{155}$$

However, because the bicontinuity constant arising in Lemma 14 does not depend on T and the condition (corresponding to (144) in the definition of $\mathcal{K}_q$)

$$\lim_{T \to 0} \Big\| \sup_{0 < t < T} |S(t)v_0| \Big\|_3 = 0 \tag{156}$$

is *not verified* if $v_0 \in L^3$, $v_0 \not\equiv 0$, there is no evidence to guarantee that such a global solution is strongly continuous at the origin (and thus unique as we will see in the following pages), and, which is intimately related, that such a solution exists locally in time for an arbitrary initial data v_0 in L^3.

We use here the same trick introduced in [34]. More precisely, instead of looking for a mild solution $v(t, x) \in \mathcal{M}$, via the point fixed Lemma 4, we will look for a solution

$$w(t, x) = v(t, x) - S(t)v_0 \in \mathcal{M} \tag{157}$$

via the point fixed Lemma 5. More precisely, we will solve the equation

$$w(t, x) = \widetilde{B}\big(S(t)v_0, S(t)v_0\big) + 2\widetilde{B}\big(w, S(t)v_0\big) + \widetilde{B}(w, w), \tag{158}$$

where the symmetric bilinear operator $\widetilde{B}$ is defined, in terms of B, by

$$\widetilde{B}(v, u)(t) = \frac{B(v, u)(t) + B(u, v)(t)}{2}. \tag{159}$$

We can now take advantage of the particular structure of the heat semigroup appearing in (158). More exactly, we can generalize the previous lemmata and obtain the following ones:

LEMMA 16. *Let $\alpha = 1 - 3/q$ and $3 < q < \infty$ be fixed. Then*

$$\Big\| \sup_{0 < t < T} t^{\alpha/2} |S(t)v_0| \Big\|_q \leqslant C_q \|v_0\|_3, \tag{160}$$

and in particular, if $v_0 \in L^3$, the left-hand side of (160) *tends to zero as T tends to zero.*

Now $\alpha > 0$, so (160) is a direct consequence of Proposition 4 and the following Sobolev embedding (see [225,226])

$$L^3 = \dot{F}_3^{0,2} \hookrightarrow \dot{F}_q^{-\alpha,2} \hookrightarrow \dot{F}_q^{-\alpha,\infty}. \tag{161}$$

LEMMA 17. *Let* $\alpha = 1 - 3/q$, $3 < q < 6$, *and* $f(t, x) = S(t) f_0$, *with* $f_0 = f_0(x)$, *then the following estimate holds for the bilinear operator*

$$\left\| B\big(S(t) f_0, S(t) f_0\big) \right\|_{\mathcal{M}} \leqslant C_q \left\| \sup_{0 < t < T} t^{\alpha/2} \big| S(t) f_0 \big| \right\|_q^2. \tag{162}$$

LEMMA 18. *Let* $\alpha = 1 - 3/q$, $3 < q < \infty$, *and* $f(t, x) = S(t) f_0$, *with* $f_0 = f_0(x)$, *and* $g = g(t, x)$ *then the following estimate holds for the bilinear operator*

$$\left\| B\big(S(t) f_0, g\big) \right\|_{\mathcal{M}} \leqslant C_q' \|g\|_{\mathcal{M}} \left\| \sup_{0 < t < T} t^{\alpha/2} \big| S(t) f_0 \big| \right\|_q. \tag{163}$$

We can now state the following existence and uniqueness theorem of [34,47] as:

THEOREM 4. *Let* $3 < q < 6$ *and* $\alpha = 1 - 3/q$. *There exists a constant* $\delta_q > 0$ *such that, for any initial data* $v_0 \in L^3$, $\nabla \cdot v_0 = 0$ *in the sense of distributions such that*

$$\left\| \sup_{0 < t < T} t^{\alpha/2} \big| S(t) v_0 \big| \right\|_q < \delta_q, \tag{164}$$

then there exists a mild solution $v(t, x)$ *belonging to* $\mathcal{N}$, *which tends strongly to* v_0 *as time goes to zero. Moreover, this solution belongs to the space* $\mathcal{M}$ *and the function* $w(t)$ *defined in* (157) *belongs to* $\mathcal{H}_p^{3/p-1}$ ($3/2 < p < 3$). *In particular,* (164) *holds for arbitrary* $v_0 \in L^3$ *provided we consider* $T(v_0)$ *small enough, and as well if* $T = \infty$, *provided the norm of* v_0 *in the Triebel–Lizorkin space* $\dot{F}_q^{-\alpha, \infty}$ *is smaller than* δ_q.

The existence part of the proof is now a consequence of Lemma 5, while its uniqueness will be treated in Section 5.3.

In order to appreciate the result we have just stated, let us now concentrate on comparing the hypotheses that arise in the statements of Theorems 3 and 4.

It is not difficult to see that, for any $T > 0$ and $3 \leqslant q \leqslant \infty$, $\alpha = 1 - 3/q$,

$$\sup_{0 < t < T} t^{\alpha/2} \big\| S(t) v_0 \big\|_q \leqslant \left\| \sup_{0 < t < T} t^{\alpha/2} \big| S(t) v_0 \big| \right\|_q \tag{165}$$

which corresponds, for $T = \infty$, to the well-known embedding

$$\dot{F}_q^{-\alpha, \infty} \hookrightarrow \dot{B}_q^{-\alpha, \infty}. \tag{166}$$

This circumstance indicates that, as far as the initial data v_0 is concerned, condition (146) is stronger than (164). However, with regard to the Navier–Stokes equations in the presence of a nontrivial external force (e.g., the gravity) as described in (11) with $\phi \not\equiv 0$, Calderón's method allows us to obtain some better estimates, in particular, as explained in [47], to improve the results contained in [56].

Before ending this section, we would like to remark that the idea of interchanging time and space in the mixed norms can also be adapted in the case of different spaces for the

Navier–Stokes equations. Explicit calculations were performed in [34] in the case of the above defined Sobolev-type space $\mathcal{H}_2^s$ ($s \geqslant 1/2$). In fact, Lemma 15 would be enough to derive such a result when $s = 1/2$. However, other less trivial examples can be obtained.

3.4.3. *Giga's space.* As we recalled in the previous section, the method for finding a strongly continuous solution with values in L^3 makes use of an *ad hoc* auxiliary subspace of functions that are continuous in the t-variable and take values in a Lebesgue space in the x-variable. Moreover, Giga proved in [99] that not only does the solution under consideration belong to $L_t^\infty(L_x^3)$ and $\mathcal{K}_q$ but also, for all q in the interval $3 < q \leqslant 9$, it belongs to the space $\mathcal{G}_q = L_t^{2/\alpha}(L_x^q)$, whose elements $f(t, x)$ are such that

$$\|f\|_{\mathcal{G}_q} =: \left(\int_0^T \|f(t, x)\|_q^{2/\alpha} \, \mathrm{d}t \right)^{\alpha/2} < \infty, \tag{167}$$

T being, as usual, either finite or infinite, and $\alpha = \alpha(q) = 1 - 3/q$.

At this point, one can naturally ask whether these spaces $\mathcal{G}_q$ can be used, independently, as auxiliary *ad hoc* subspaces to prove the existence of a solution with data in L^3. This question arises also in view of the fact that $L_t^p(L_x^q)$ estimates (and, more generally, the so-called Strichartz estimates) are frequently used for the study of other well-known nonlinear partial differential equations, like the Schrödinger one or the wave equation. Even if this does not lead here to a breakthrough as in the case of the Schrödinger equation, making *direct* use of $L_t^p(L_x^q)$ estimates for Navier–Stokes is indeed possible. This was proved by Kato and Ponce in [118], where, in fact, the authors consider the case of a much larger functional class, including the $\mathcal{G}_q$ one.

In what follows, we will focus our attention only on the latter case and prove an existence theorem of local (resp. global) strong solutions in $\mathcal{C}([0, T); L^3)$ with initial data (resp. small enough) in a certain Besov space.

The "Besov language" will provide a very convenient and powerful tool, needed to overcome difficulties which were absent in the previous section.

As in the previous cases, we will start with an estimate of the linear term $S(t)v_0$ in the auxiliary space $\mathcal{G}_q$. We have following lemma.

LEMMA 19. *Let $3 < q \leqslant 9$ and $\alpha = 1 - 3/q$ be fixed. Then*

$$\left(\int_0^T \|S(t)v_0\|_q^{2/\alpha} \, \mathrm{d}t \right)^{\alpha/2} \leqslant C_q \|v_0\|_3, \tag{168}$$

where the integral in the left-hand side of (168) *tends to zero as T tends to zero provided $v_0 \in L^3$.*

Keeping Proposition 4 in mind, this lemma can be proved if we recall the well-known Sobolev embedding [225,226]

$$L^3 \hookrightarrow \dot{B}_q^{-\alpha, 2/\alpha}, \tag{169}$$

which holds true as long as $3 < q \leqslant 9$. Here the restriction $q \leqslant 9$ appears as a limit exponent in the Sobolev embedding for Besov spaces. A direct proof of (168) is contained in the papers by Giga [99], Kato [114] and Kato and Ponce [118] and makes use of the Marcinkiewicz interpolation theorem. In short, our lemma says that if $v_0 \in L^3$, then $S(t)v_0$ is in $\mathcal{G}_q$, and therefore we are allowed to work within that functional framework.

The fact that the left-hand side of (168) tends to zero as T tends to zero can be easily checked by using the Banach–Steinhaus theorem. What we would like to stress here, is that this property is of paramount importance, because it will ensure (as in Theorems 3 and 4) the strong continuity at the origin of the solution given by the fixed point scheme. Once we get a solution in $\mathcal{C}([0, T); L^3)$ that tends in the strong L^3 topology to v_0 as time tends to zero, this solution will automatically be *unique*, as we will see in Section 5.3.

Let us now concentrate on the bilinear operator [186].

LEMMA 20. *The bilinear operator $B(f, g)(t)$ is bicontinuous from $\mathcal{G}_q \times \mathcal{G}_q \to \mathcal{G}_p$ for any $3 < p < \frac{3q}{6-q}$ if $3 < q < 6$; any $3 < p < \infty$ if $q = 6$; and $q/2 \leqslant p \leqslant \infty$ if $6 < q < \infty$.*

In the case $q = p$ this result was originally proved by Fabes, Jones and Rivière [76] and represents the equivalent of Lemma 10 in the space $\mathcal{K}_q$.

This lemma can be proved by duality (in the t-variable) in a way reminiscent of Giga's method introduced in [99] and based on the Hardy–Littlewood–Sobolev inequality (see [47]). Here the restrictions on the exponents p and q come from the Young and Hardy–Littlewood–Sobolev inequalities. In particular, the value $\beta = 0$ corresponding to $p = 3$ is excluded. This is why Lemma 20 *cannot be used* directly to get (as in Lemma 11) an $L_t^\infty(L_x^3)$ estimate. That appears to be the main difference with the methods involving the Besov $\dot{B}_q^{-(1-3/q),\infty}$ and Triebel–Lizorkin spaces $\dot{F}_q^{-(1-3/q),\infty}$ that were considered in the previous cases. As a matter of fact, the estimates obtained in those spaces, having their third index equal to ∞, are essentially based on the scaling invariance of the Navier–Stokes equations, which is a very crude property of the nonlinear term. Here, on the contrary, we need to investigate further and to explicitly take into account the oscillatory property of the bilinear term, say

$$\int_{\mathbb{R}^3} \Theta(x)\,\mathrm{d}x = 0 \tag{170}$$

or, equivalently, the fact that the Fourier transform of Θ is zero at the origin. Of course, we are still far away from exploiting the full structure of the bilinear term.

This remark being made, let us now see how to use (170) in the proof of the following lemma.

LEMMA 21. *The bilinear operator $B(f, g)(t)$ is bicontinuous from $\mathcal{G}_6 \times \mathcal{G}_6 \to \mathcal{N}$. In fact, $B(f, g)$ takes its values in $\mathcal{C}(0, T; \dot{B}_3^{0,2})$, which is a proper subset of $\mathcal{N}$.*

We would like to mention here that a variant of this result was applied in [91,92] in the proof of the uniqueness theorem for strong L^3 solutions (see also [48] for more comments) as we will see in Section 5.3.

Let us now outline the proof of Lemma 21, by using once again a duality argument: first we show that $B(f, g)$ is bicontinuous from $L_t^4(L_x^6) \times L_t^4(L_x^6)$ into $L_t^\infty(\dot{B}_3^{0,2})$ and then we conclude by a usual argument in order to restore the strong continuity in time [47].

To prove the proposition by duality (in the x-variable), let us consider an arbitrary test function $h(x) \in \mathcal{C}_0^\infty$ and let us evaluate

$$I_t = \int_{\mathbb{R}^3} \int_0^t (t - s)^{-2} \Theta\left(\frac{\cdot}{\sqrt{t - s}}\right) * (fg)(s)h(x) \, ds \, dx. \tag{171}$$

It is useful here to see the t variable as a fixed parameter. After interchanging the integral over $\mathbb{R}^3$ with the convolution with $h(x)$, and after applying the Hölder inequality (in x) and the Cauchy–Schwarz inequality (in t), we get

$$|I_t| \lesssim \left(\int_0^t \|fg\|_3^2 \, ds\right)^{1/2} \left(\int_0^t \|\Theta_u * h\|_{3/2}^2 \, \frac{du}{u}\right)^{1/2}, \tag{172}$$

where

$$\Theta_u = \frac{1}{u^3} \Theta\left(\frac{\cdot}{u}\right). \tag{173}$$

In order to conclude, we only remark that the oscillatory property of Θ, say (170), allows us to consider the quantity

$$\left(\int_0^\infty \|\Theta_u * h\|_{3/2}^2 \, \frac{du}{u}\right)^{1/2} \tag{174}$$

as an (equivalent) norm on the homogeneous Besov space $\dot{B}_{3/2}^{0,2}$. As we observed in Section 2.2, if the function Θ were smooth and compactly supported on the Fourier side, this would indeed be the usual characterization. Removing the band-limited condition is trivial, and it turns out that smoothness is not a critical assumption, thus allowing a greater flexibility in the definition of the Besov space. What is certainly *not possible* is to get such an equivalence if, as is the case for $S(t)$, the function Θ does not have a zero integral. More explicitly, a property analogous to the one stated in Proposition 3 would not apply here and, in general, does not apply for a Besov space of the type $\dot{B}_q^{s,p}$, with $s \geqslant 0$. A counterexample for $s = 0$, $p = \infty$ and $q = 3$ can be found, for instance, in [34] (Lemma 4.2.10). The reader should refer to [185] for a very enlightening discussion of the definition of Besov spaces, and to [82,225,226] for precise results.

Let us go back to the Besov space $\dot{B}_{3/2}^{0,2}$. A standard argument shows that the dual space of $\dot{B}_{3/2}^{0,2}(\mathbb{R}^3)$ is exactly $\dot{B}_3^{0,2}$. All this finally implies that the bilinear operator $B(f, g)$ is bicontinuous from $L_t^4(L_x^6) \times L_t^4(L_x^6)$ into $L_t^\infty(\dot{B}_3^{0,2})$, which completes the proof of Lemma 21. Moreover, as

$$\dot{B}_3^{0,2} \hookrightarrow L^3, \tag{175}$$

we have obtained our $L_t^\infty(L_x^3)$ estimation, and even improved it. As in [34,43,48,186], this provides another example in which the regularity of the bilinear term is better than the linear one.

We are now in position to prove the following theorem [186]:

THEOREM 5. *Let $3 < q < 9$ and $\alpha = 1 - 3/q$ be fixed. There exists a constant δ_q such that for any initial data $v_0 \in L^3$, $\nabla \cdot v_0 = 0$ in the sense of distributions such that*

$$\left(\int_0^T \left\| S(t)v_0 \right\|_q^{2/\alpha} \, \mathrm{d}t \right)^{\alpha/2} < \delta_q \tag{176}$$

and then there exists a mild solution $v(t, x)$ belonging to $\mathcal{N}$, which tends strongly to v_0 as time goes to zero. Moreover, this solution belongs to all the spaces $\mathcal{G}_q$ ($3 < q < 9$) and is such that the fluctuation $w(t, x)$ defined in (157) satisfies

$$w \in \mathcal{C}\big([0, T); \dot{B}_3^{0,2}\big) \tag{177}$$

and

$$w \in L^2\big((0, T); L^\infty\big). \tag{178}$$

Finally, (176) *holds for arbitrary $v_0 \in L^3$ provided we consider $T(v_0)$ small enough, and as well if $T = \infty$, provided the norm of v_0 in the Besov space $\dot{B}_q^{-\alpha,2/\alpha}$ is smaller than δ_q.*

Keeping in mind the previous propositions and remarks, the proof of that theorem is easily carried out as follows (see [186] for more details).

First, we apply the fixed point algorithm in the space $\mathcal{G}_q = L^{2/\alpha}([0, T); L^q)$ (q and α being assigned in the statement) to get, by means of Lemma 20, a mild solution $v(t, x) \in \mathcal{G}_q$. Then, again using Lemma 20, we find that $v(t, x) \in \mathcal{G}_q$ for all $3 < q < 9$. In particular $v(t, x) \in \mathcal{G}_6 = L^4([0, T); L^6)$, which gives $v(t, x) \in \mathcal{N}$ and (177) (once Lemma 21 is taken into account).

As we presented in [48], this regularity result can even be improved to get $w(t) \in \mathcal{C}([0, T); \dot{F}_{3/2}^{1,2})$, which means that the gradient of $w(t)$ belongs uniformly in time to $L^{3/2}$ and we observe that $\dot{F}_{3/2}^{1,2} \hookrightarrow \dot{B}_3^{0,2}$. The latter regularity result can be seen in connection with an estimate derived by Kato [114] that assures that the gradient of $v(t)$, solution of Theorem 3 in $\mathcal{N}$, is such that $t^{1-3/(2q)}\nabla v(t) \in \mathcal{C}([0, T); L^q)$ for any $q \geqslant 3$. We proved in [48] that the function $w(t)$ satisfies the last estimate for the optimal exponent $q = 3/2$.

Finally, as the bilinear term is bicontinuous from $\mathcal{G}_q \times \mathcal{G}_q$ into $L_t^2(L_x^\infty)$, and arguing by duality, ($\mu(s)$ being a test function), we can obtain the estimate (178), say

$$\left| \int_0^T \left\| B(f, g)(s) \right\|_\infty \mu(s) \, \mathrm{d}s \right| \lesssim \int_0^T \int_0^t \frac{\|fg\|_{q'/2}(s)\mu(t)}{(t-s)^{1/2+3/q'}} \, \mathrm{d}s \, \mathrm{d}t \lesssim \|\mu\|_2. \tag{179}$$

4. Highly oscillating data

At difference with Leray's well-known weak approach, the method described in the previous pages – the so-called "Tosio Kato's method" (see the book [88] for many examples of applications of this method to nonlinear PDEs) – also implies the uniqueness of the corresponding solution, as it will be explained in Section 5. However, the existence of the solution holds under a restrictive condition on the initial data, that is required to be small, which is not the case for Leray's weak solutions. In Section 7 we will make the link between this property, the smallness of the Reynolds number associated with the flow, the stability of the corresponding global solution and the existence of Lyapunov functions for the Navier–Stokes equations.

The aim of this section is to give an interpretation of the smallness of the initial data in terms of an oscillation property. The harmonic analysis tools we developed so far will play a crucial role here.

Let us recall that, as stated in Theorem 3, a global solution in $\mathcal{C}([0, \infty); L^3)$ exists, provided that the initial data v_0 is divergence-free and belongs to L^3, and that its norm is small enough in L^3, or more generally, small in the Besov space $\dot{B}_q^{-\alpha, \infty}$ (for a certain $3 < q < \infty$ and $\alpha = 1 - 3/q$ fixed). In other words, a function v_0 in L^3 whose norm is arbitrarily large in L^3 but small enough in $\dot{B}_q^{-\alpha, \infty}$ (or in a Triebel–Lizorkin space $\dot{F}_q^{-\alpha, \infty}$ as in Theorem 4, or in the Besov space $\dot{B}_q^{-\alpha, 2/\alpha}$ as in Theorem 5) also ensures the existence of a global mild solution in $\mathcal{C}([0, \infty); L^3)$.

The advantage of using a Besov norm instead of a Lebesgue one is that the condition of being small enough in a Besov space is satisfied by highly oscillating data (Section 4.1). A second remarkable property is that these spaces contain homogeneous functions of degree -1, leading to global self-similar solutions (Section 6). Moreover, Besov spaces led to the (first) proof of the uniqueness for solutions in $\mathcal{C}([0, \infty); L^3)$ (Section 5.3).

The a posteriori disappointing observation is that ... Besov spaces were not necessary at all in any of these discoveries!

4.1. *A remarkable property of Besov spaces*

In order to appreciate the formulation of Kato's theorem in terms of the Besov space $\dot{B}_q^{-\alpha, \infty}$ given in Theorem 3, we shall devote ourselves here to illustrating that the condition $\|v_0\|_{\dot{B}_q^{-\alpha, \infty}} < \delta$ is satisfied in the particular case of a sufficiently oscillating function v_0.

A typical situation will be given by the following example. Let v_0 be an arbitrary (not identically vanishing) function belonging to L^3. If we multiply v_0 by an exponential, say the function $w_k = \exp[ix \cdot k]$, we obtain, for any $k \in \mathbb{R}^3$, a function $w_k v_0$ such that (Lemma 22)

$$\lim_{|k| \to \infty} \|w_k v_0\|_{\dot{B}_q^{-\alpha, \infty}} = 0, \tag{180}$$

in spite of the fact that

$$\lim_{|k| \to \infty} \|w_k v_0\|_3 = \|v_0\|_3. \tag{181}$$

In other words, the smallness condition $\|w_k v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$, is verified as long as we choose a sufficiently high frequency k. At this point, it is tempting to consider $w_k v_0$ as the new initial data of the problem and to affirm that Kato's solution exists globally in time, provided we consider sufficiently oscillating data. One can argue that $w_k v_0$ is no longer a divergence-free function. Nevertheless, the function $w_k v_0$ is divergence-free *asymptotically* for $|k| \to \infty$, which is exactly the situation we are dealing with. More precisely, it turns out that (Lemma 23)

$$\lim_{|k|\to\infty} \left\| \nabla \cdot (w_k v_0) - w_k \nabla \cdot v_0 \right\|_3 = 0. \tag{182}$$

LEMMA 22. *Let v be an arbitrary function in L^3 and let $w_k(x)$, $k \in \mathbb{N}$, be a sequence of functions such that $\|w_k\|_\infty \leqslant C$ and $w_k \rightharpoonup 0$ (as $k \to \infty$) in the distributional sense. Then, the products $w_k v$ tend to 0 in the strong topology of $\dot{B}_q^{-\alpha,\infty}$ ($\alpha = 1 - 3/q > 0$).*

The proof of this lemma is quite easy and we wish to present the main components here (for more details see [34,35]).

We will make use of a density argument. To this end, let us introduce the following decomposition of the function v:

$$v = h + g, \tag{183}$$

where $h \in L^3$ and

$$\|h\|_3 \leqslant \varepsilon \tag{184}$$

and $g \in \mathcal{C}_0^\infty$. The next step is to recall the continuous embedding (Lemma 9) $L^3 \hookrightarrow \dot{B}_q^{-\alpha,\infty}$ to infer the following inequality ($k \geqslant 0$)

$$\|w_k h\|_{\dot{B}_q^{-\alpha,\infty}} \lesssim \|w_k h\|_3 \lesssim \varepsilon. \tag{185}$$

On the other hand, Young's inequality gives ($j \in \mathbb{Z}$)

$$\left\| S_j(w_k g) \right\|_q \leqslant \left\| 2^{3j} \varphi(2^j \cdot) \right\|_r \|w_k g\|_p, \tag{186}$$

where

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1. \tag{187}$$

This implies

$$2^{-\alpha j} \left\| S_j(w_k g) \right\|_q \lesssim 2^{-j(1-3/q)} 2^{-j(1-3/r)} \|g\|_p = 2^{-j(1-3/p)} \|g\|_p \tag{188}$$

so that, for any $k \geqslant 0$, any $j \geqslant j_1 > 0$ and any $j \leqslant j_0 < 0$, we have

$$2^{-\alpha j} \left\| S_j(w_k g) \right\|_q \lesssim \varepsilon \tag{189}$$

(in fact, if $j \geqslant j_1$ we let $p = q > 3$ and if $j \leqslant j_0$ we let $1 \leqslant p < 3$).

We are now left with the terms $S_j(w_k g)$ for $j_0 < j < j_1$. Making use of the hypothesis $m_k \rightharpoonup 0$ together with the Lebesgue dominated convergence theorem, we finally find, for any $k \geqslant k_0$ and $j_0 < j < j_1$,

$$2^{-\alpha j} \left\| S_j(w_k g) \right\|_q \lesssim \varepsilon \tag{190}$$

which concludes the proof of the lemma.

LEMMA 23. *Let $m(\xi) \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\})$ be a homogeneous function of degree 0 and let M be the convolution operator associated with the multiplier $m(\xi)$. If we consider $|\xi_0| = 1$, $v \in L^p$ and $1 < p < \infty$, then*

$$\lim_{\lambda \to \infty} \sup_{|\xi_0|=1} \left\| M\big(\exp(\mathrm{i}\lambda\xi_0 \cdot x)v(x)\big) - \exp(\mathrm{i}\lambda\xi_0 \cdot x)m(\xi_0)v(x) \right\|_p = 0. \tag{191}$$

In the case we are interested in, this lemma will be used for $p = 3$ and with M replaced by the Leray–Hopf projection operator $\mathbb{P}$ onto the divergence-free vector fields and $m(\xi)$ replaced by a 3×3 matrix whose entries are homogeneous symbols of degree 0.

In order to prove the lemma in its general form, we remark that the symbol of the operator $\exp(-\mathrm{i}\lambda\xi_0 \cdot x)M(\exp(\mathrm{i}\lambda\xi_0 \cdot x)v) - m(\lambda\xi_0)v(x)$ is given by $m(\xi + \lambda\xi_0) - m(\lambda\xi_0)$, this by virtue of the homogeneity of m.

Equation (191) will now be proved by means of a density argument. In fact, it is sufficient to limit ourselves to functions $v \in \mathcal{V} \subset L^p$, where $\mathcal{V}$ is the dense subspace of L^p defined by $v \in \mathcal{S}$ and the Fourier transform $\hat{v}$ of v has compact support. Now, we put

$$v_\lambda = \exp(-\mathrm{i}\lambda\xi_0 \cdot x)M\big(\exp(\mathrm{i}\lambda\xi_0 \cdot x)v\big) - m(\lambda\xi_0)v, \tag{192}$$

then the Fourier transform of v_λ is given by

$$\hat{v}_\lambda(\xi) = \big[m(\xi + \lambda\xi_0) - m(\lambda\xi_0)\big]\hat{v}(\xi). \tag{193}$$

Finally, $\hat{v}$ has compact support, say in $|\xi| \leqslant R$, and then

$$m(\xi + \lambda\xi_0) - m(\lambda\xi_0) = r_\lambda(\xi), \tag{194}$$

where, on $|\xi| \leqslant R$, $r_\lambda(\xi) \to 0$ together with all its derivatives in the L^∞ norm. We thus have $v_\lambda \to 0$ in $\mathcal{S}$ when $\lambda \to \infty$. A fortiori, $\|v_\lambda\|_p \to 0$ when $\lambda \to \infty$, and the lemma is proved.

4.2. *Oscillations without Besov norms*

Some years after the publication of [34,35] Temam [217] informed us that the property we described in the previous pages, that highly oscillating data lead to global

solutions to Navier–Stokes, was implicitly contained in the pioneering papers of Kato and Fujita [87,117] of 1962.

These papers deal with mild solutions to Navier–Stokes that are continuous in time and take values in the Sobolev space $\dot{H}^s$, say $v \in \mathcal{C}([0, T); \dot{H}^s)$. It is easy to see, in the three-dimensional case, that the critical Sobolev space corresponds to the value $s = 1/2$. More precisely, the Sobolev spaces $\dot{H}^s$, $s > 1/2$ are super-critical. In other words, as far as the scaling is concerned, they have the same invariance as the Lebesgue spaces L^p if $p > 3$. This means that, using the simplified version of the bilinear operator, one can easily prove the existence of a local mild solution for arbitrary initial data [34], that is, the theorem.

THEOREM 6. *Let $1/2 < s < \infty$ be fixed. For any $v_0 \in \dot{H}^s$, $\nabla \cdot v_0 = 0$, there exists a $T = T(\|v_0\|_s)$ such that the Navier–Stokes equations have a mild solution in $\mathcal{C}([0, T); \dot{H}^s)$.*

On the other hand, in the critical case $s = 1/2$, one can ensure the existence of a local solution, that turns out to be global when the initial data are small enough:

THEOREM 7. *There exists a constant $\delta > 0$ such that for any initial data $v_0 \in \dot{H}^{1/2}$, $\nabla \cdot v_0 = 0$ in the sense of distributions, such that*

$$\|v_0\|_{\dot{H}^{1/2}} < \delta, \tag{195}$$

then there exists a mild solution $v(t, x)$ to the Navier–Stokes equations belonging to $\mathcal{C}([0, \infty); \dot{H}^{1/2})$.

In the particular case $s = 1$, we also have at our disposal a persistence result, namely:

THEOREM 8. *There exists a constant $\delta > 0$ such that if the initial data $v_0 \in \dot{H}^{1/2} \cap \dot{H}^1$, $\nabla \cdot v_0 = 0$ in the sense of distributions and satisfies*

$$\|v_0\|_{\dot{H}^{1/2}} < \delta, \tag{196}$$

then the mild solution $v(t, x)$ to the Navier–Stokes equations, whose existence is ensured by Theorem 7, also belongs to $\mathcal{C}([0, \infty); \dot{H}^1)$.

To prove such a result, it is enough to show that the $\dot{H}^1$ norm of the solution is a Lyapunov function, which means that it is decreasing in time. The study of the Lyapunov functions for the Navier–Stokes equations will be examined in detail in Section 7.1.

Actually, to obtain a global mild solution in the space $\mathcal{C}([0, \infty); \dot{H}^1)$ it would be enough to get a uniform estimate of the kind

$$\|v(t)\|_{\dot{H}^1} \leqslant \|v_0\|_{\dot{H}^1} \quad \forall t > 0, \tag{197}$$

because a classical "bootstrap" argument will allow to pass from a local solution to a global one.

This property turns out to be satisfied when the initial data $v_0 \in \dot{H}^1$ has a sufficiently small norm in the space $\dot{H}^{1/2}$. More precisely, as we will describe in detail in Section 7.1, the following inequality is proven in the celebrated papers by Kato and Fujita [87,117]:

$$\frac{\mathrm{d}}{\mathrm{d}t} \| v(t) \|_{\dot{H}^1}^2 \leqslant -2 \| v(t) \|_{\dot{H}^2}^2 \big(\nu - C \| v(t) \|_{\dot{H}^{1/2}} \big). \tag{198}$$

This immediately implies the aforementioned property of decrease in time of the homogeneous norm $\|v\|_{\dot{H}^1}$, as long as $\|v_0\|_{\dot{H}^{1/2}}$ is small enough. On the other hand, it is easy to show that the L^2 norm of the solution v also decreases in time, say

$$\frac{\mathrm{d}}{\mathrm{d}t} \| v(t) \|_2^2 = -2\nu \| \nabla v(t) \|_2^2 < 0, \tag{199}$$

which allows us to deduce the decreasing of the nonhomogeneous norm $\|v\|_{H^1}$ as well.

Now, Temam's remark is very simply and reads as follows. Suppose $v_0 \in \mathcal{S}'$ is such that $\hat{v}_0(\xi) = 0$ if $|\xi| \leqslant R$, then

$$\|v_0\|_{\dot{H}^{1/2}} \leqslant R^{-1/2} \|v_0\|_{\dot{H}^1} \tag{200}$$

and thus one can get the existence of a global mild solution in $\mathcal{C}([0, \infty); \dot{H}^1)$ provided the initial data is concentrated at high frequencies ($R \gg 1$), say highly oscillating!

4.3. *The result of Koch and Tataru*

In his doctoral thesis [186,187], Planchon gave the precise interpretation of the persistence result stated in Theorem 8, replacing the smallness of the $\dot{H}^{1/2}$ norm of the initial data, with the smallness (or oscillation) in a Besov space. Everything takes place as in [34] for the critical space L^3: there exists an absolute constant $\beta > 0$ such that if $\|v_0\|_{\dot{B}_4^{-1/4,\infty}} < \beta$ and $v_0 \in \dot{H}^1$, then there exists a global solution in $\mathcal{C}([0, \infty); \dot{H}^1)$. What make things work here is that, even if $\dot{H}^1$ is not a critical space, it is embedded in $\dot{H}^{1/2}$ (which is not the case for any Lebesgue space L^p, $p \geqslant 3$, when working in unbounded domains as $\mathbb{R}^3$). The importance of such a result is that it allows us to obtain *global* and *regular* solutions in the energy space $\dot{H}^1$, under the hypothesis of oscillation of the initial data. In other words, at variance with the L^3 setting, we can establish a link between Leray's weak solutions and Kato's mild ones.

This approach was generalized first by Koch and Tataru [123] and then by Furioli, Lemarié, Zahrouni and Zhioua [89,93,145,240]. Both of these results seem optimal.

Roughly speaking the theorem by Koch and Tataru says that if the norm of the initial data is small enough in the critical space BMO^{-1}, then there is a global mild solution for the Navier–Stokes equations. Again, the norm of the product of a fixed function in L^3 times an oscillating function, say $w_k = \exp[ix \cdot k]$, tends to zero as $|k|$ tends to infinity. It is not clear whether this theorem is optimal, because, if it is true that it generalizes the results of the previous section (in fact BMO^{-1} contains L^3 as well as $\dot{B}_q^{-\alpha,\infty}$, for any $3 < q < \infty$ and $\alpha = 1 - 3/q$), we should recall that BMO^{-1} is contained in the biggest

critical space $\dot{B}_\infty^{-1,\infty}$ (as stated in (77) and Proposition 7) and nobody knows whether the Navier–Stokes system is well posed in this space (see [166]). Incidentally, we wish to remind the reader that Montgomery-Smith proved a blow-up result in the space $\dot{B}_\infty^{-1,\infty}$ for a modified (with respect to the nonlinear term) Navier–Stokes equations [176]. Moreover, his result also shows there is initial data that exists in every Triebel–Lizorkin or Besov space (and hence in every Lebesgue and Sobolev space), such that after a finite time, the solution of the Navier–Stokes-like equation is in no Triebel–Lizorkin or Besov space (and hence in no Lebesgue or Sobolev space).

On the other hand, the persistence result by Furioli, Lemarié, Zahrouni and Zhioua says that if the initial data is not only small in BMO^{-1}, but also belongs to the Banach space X, where X can be either the Lebesgue space L^p, $1 \leqslant p \leqslant \infty$, or the inhomogeneous Besov space $B_q^{s,p}$ with $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and $s > -1$, or the homogeneous Besov space $\dot{B}_q^{s,p}$ with $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and $s > -1$, then the corresponding solution also belongs to $L^\infty((0, \infty); X)$.

In order to simplify the discussion, we will limit ourselves to present only global solutions. However, solutions which are local in time as we previously constructed in the critical space L^3 are also available. More exactly, we are talking about the following results.

THEOREM 9. *There exists a constant $\delta > 0$ such that, for any initial data $v_0 \in BMO^{-1}$ that verifies*

$$\|v_0\|_{BMO^{-1}} < \delta, \tag{201}$$

then there exists a global mild solution $v(t, x)$ to the Navier–Stokes equations such that

$$\sqrt{t}\,v(t, x) \in L^\infty\big((0, \infty), \mathbb{R}^3\big) \tag{202}$$

and

$$\sup_{t>0, x_0 \in \mathbb{R}^3} \frac{1}{t^{3/2}} \int_{0<\tau<t} \int_{|x-x_0|<\sqrt{t}} \big|v(\tau, x)\big|^2 \, \mathrm{d}\tau \, \mathrm{d}x < \infty. \tag{203}$$

The proof of this theorem is contained in the paper of Koch and Tataru [123]. The condition expressed by (203), comes from the fact that a Carleson measure characterization of BMO^{-1} (see [214]) says that a function v_0 belongs to BMO^{-1} if and only if

$$\sup_{t>0, x_0 \in \mathbb{R}^3} \frac{1}{t^{3/2}} \int_{0<\tau<t} \int_{|x-x_0|<\sqrt{t}} \big|S(\tau)v_0\big|^2 \, \mathrm{d}\tau \, \mathrm{d}x < \infty, \tag{204}$$

$S(\tau) = \exp(\tau \Delta)$ denoting, as usual, the heat semigroup. On the other hand, this condition seems the weaker possible one, say BMO^{-1} seems the largest space where local or global solutions exist. In fact, as we recalled in Section 1.2, in order to give a sense to the Navier–Stokes equations we want to have at least

$$v(t, x) \in L^2_{\text{loc}}\big([0, \infty); \mathbb{R}^3\big). \tag{205}$$

Now the Navier–Stokes equations are invariant with respect to scaling, hence we want a scale and translation invariant version of L^2-boundedness, say

$$\sup_{t>0,x_0\in\mathbb{R}^3} \frac{1}{|B_t(x)|} \iint_{B_t(x)\times[0,t^2]} \left|v(\tau,x)\right|^2 \mathrm{d}\tau\,\mathrm{d}x < \infty \tag{206}$$

(where $|B_t(x)|$ denotes the Lebesgue measure of the ball $B_t(x)$ centered at x and radius t), which is precisely the condition expressed by (203).

Finally, let us quote the persistence result announced in [93].

THEOREM 10. *Let v_0 verify the condition of Theorem 9 and $v(t,x)$ the corresponding global solution, then if X is one of the following Banach spaces*:

$$\text{Lebesgue } L^p, \quad 1 \leqslant p \leqslant \infty, \tag{207}$$

or

$$\text{inhomogeneous Besov } B_q^{s,p}, \quad 1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty, s > -1, \tag{208}$$

or

$$\text{homogeneous Besov } \dot{B}_q^{s,p}, \quad 1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty, s > -1, \tag{209}$$

then the corresponding solution also belongs to $L^\infty((0,\infty); X)$.

From the sketch of the proof contained in [93] it is clear that this result applies more generally to any Banach space X such that the following condition is satisfied

$$\|fg\|_X \lesssim \left(\|f\|_X \|g\|_\infty + \|g\|_X \|f\|_\infty \right), \tag{210}$$

as is the case for the spaces quoted above as well as for the Sobolev space H^s, $s \geqslant 1/2$.

5. Uniqueness theorems

In 1994 Jean Leray summarized the state of the art for the Navier–Stokes equations in the following way [150]:

> A fluid flow initially regular remains so over a certain interval of time; then it goes on indefinitely; but does it remain regular and well-determined? We ignore the answer to this double question. It was addressed sixty years ago in an extremely particular case [149]. At that time H. Lebesgue, questioned, declared: "Don't spend too much time for such a refractory question. Do something different!"

This is not the case for Kato's mild solutions for which a general uniqueness theorem, that is the subject of this section, is available. In order to appreciate the simplicity of its proof, let us start by recalling why the uniqueness of weak solutions remains a challenging question.

5.1. *Weak solutions*

Before dealing with the uniqueness of weak solutions for Navier–Stokes, let us examine a more general case. We consider the difference $v_1 - v_2$ of two weak solutions v_1 and v_2 that, for the moment, may take different initial values (i.e., $v_1(0) - v_2(0)$ is not necessarily zero), but with the same boundary conditions, say $v_1(t, x) - v_2(t, x) = 0$ if $x \in \partial\Omega$ for all $t > 0$ (this is always the case if we suppose the no-slip boundary conditions, $v_1 = v_2 = 0$ on $(0, T) \times \partial\Omega$)). Of course, if Ω is unbounded, this condition concerns the behavior of the solutions at infinity.

We obtain

$$\frac{\partial}{\partial t}(v_1 - v_2) + v_1 \cdot \nabla(v_1 - v_2) + (v_1 - v_2) \cdot \nabla v_2 = \Delta(v_1 - v_2) - \nabla(p_1 - p_2) \tag{211}$$

and if we take the inner product $\langle \cdot, \cdot \rangle$ of $L^2(\Omega)$ with $(v_1 - v_2)$ we finally get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_1 - v_2\|_2^2 + \left\|\nabla(v_1 - v_2)\right\|_2^2 = -\left\langle (v_1 - v_2) \cdot \nabla v_2, v_1 - v_2 \right\rangle. \tag{212}$$

In fact, since $(v_1 - v_2)(t, x) = 0$ if $x \in \partial\Omega$ for all $t > 0$, Green's formula gives

$$\begin{aligned}
&\left\langle v_1 \cdot \nabla(v_1 - v_2), v_1 - v_2 \right\rangle \\
&= -\left\langle \nabla \cdot v_1, |v_1 - v_2|^2 \right\rangle - \left\langle v_1 \cdot \nabla(v_1 - v_2), v_1 - v_2 \right\rangle = 0
\end{aligned} \tag{213}$$

and

$$\left\langle \nabla(p_1 - p_2), v_1 - v_2 \right\rangle = -\left\langle p_1 - p_2, \nabla \cdot (v_1 - v_2) \right\rangle = 0. \tag{214}$$

Thus, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_1 - v_2\|_2^2 + \left\|\nabla(v_1 - v_2)\right\|_2^2 \leqslant \|\nabla v_2\|_\infty \|v_1 - v_2\|_2^2 \tag{215}$$

which finally gives, via Gronwall's lemma, the estimate

$$\begin{aligned}
&\left\|(v_1 - v_2)(s)\right\|_2^2 + 2\int_0^s \left\|\nabla(v_1 - v_2)\right\|_2^2 \mathrm{d}t \\
&\qquad \leqslant \left\|(v_1 - v_2)(0)\right\|_2^2 \exp\left(\int_0^s 2\|\nabla v_2\|_\infty \mathrm{d}t\right)
\end{aligned} \tag{216}$$

and implies uniqueness of weak solutions as long as the (formal) manipulations we have performed are justified and the quantity $\int_0^s \|\nabla v_2\|_\infty \mathrm{d}t$ remains bounded. In particular, this argument shows the uniqueness of classical smooth solutions. More precisely, if *one* smooth weak solution, say v_2, exists and is such that $\int_0^s \|\nabla v_2\|_\infty \mathrm{d}t$ remains bounded, then *all* weak solutions have to coincide with it.

But there is another way to estimate the term $-\langle (v_1 - v_2) \cdot \nabla v_2, v_1 - v_2 \rangle$, say

$$\left| \langle (v_1 - v_2) \cdot \nabla v_2, v_1 - v_2 \rangle \right| \leqslant \|\nabla v_2\|_2 \|v_1 - v_2\|_4^2 \tag{217}$$

which suggests the use of the Sobolev inequality

$$\|v_1 - v_2\|_4 \leqslant c \|v_1 - v_2\|_2^{1-n/4} \left\| \nabla (v_1 - v_2) \right\|_2^{n/4} \tag{218}$$

where $n = 2$ or $n = 3$ denotes, as usual, the space dimension. Now, if we consider the two cases separately, we obtain after some straightforward calculations (see [110,235])

$$\left\| (v_1 - v_2)(s) \right\|_2^2 \leqslant \left\| (v_1 - v_2)(0) \right\|_2^2 \exp\!\left(c \int_0^s \|\nabla v_2\|_2^2 \, dt \right) \tag{219}$$

if $n = 2$, and

$$\left\| (v_1 - v_2)(s) \right\|_2^2 \leqslant \left\| (v_1 - v_2)(0) \right\|_2^2 \exp\!\left(c \int_0^s \|\nabla v_2\|_2^4 \, dt \right) \tag{220}$$

if $n = 3$.

If we make use of the energy inequality (21), which is the only information on weak solutions we can (and should) use here, it is easy to conclude and get a uniqueness result only in the case $n = 2$. In fact, nothing can be said if $n = 3$ because, at variance with the case $n = 2$, the energy inequality does not allow us here to treat the term $\int_0^s \|\nabla v_2\|_2^4 \, dt$. If we could, we would of course not only obtain uniqueness, but also continuous dependence on initial data and the full regularity of the solution.

A third way to obtain uniqueness was suggested by Serrin [208,209] and improved later on by many authors. The idea is that if some additional integrability property is satisfied by *at least one* weak solution, more exactly, if $v_2 \in L^s((0, T); L^r)$ and if $2/s + n/r = 1$ with $n < r \leqslant \infty$, then *all* weak solutions have to coincide with it (recently, Kozono and Taniuchi in [126] considered the marginal case $s = 2$, $r = \infty$ in a larger class, say $v_2 \in L^2((0, T); BMO)$, see also [127,194]). In general, if v_2 is a weak solution, it is possible to prove that there exist s_0 and r_0 such that $2/s_0 + n/r_0 = n/2$ so that $v_2 \in L^{s_0}((0, T); L^{r_0})$. In particular, from this remark and Serrin's criterion we can recover, in the two-dimensional case, the uniqueness result shown above. But, again, in three dimensions this is not enough to conclude.

Finally, concerning the critical exponents $n = r$ and $s = \infty$, Serrin's result was adapted by von Wahl [232] (resp. by Kozono and Sohr [125]) to obtain the following result. Suppose that *one* weak solution, say v_2, satisfies $v_2 \in \mathcal{C}([0, T); L^n)$ (resp. $v_2 \in L^\infty((0, T); L^n)$), then *all* weak solutions have to coincide with it (for a different proof see the papers of Lions and Masmoudi [152–154]). More recently, the smoothness of such a weak solution was proved by Escauriaza, Seregin and Sverák [75]. On the other hand, Montgomery-Smith announced in [177] a logarithmic improvement over the usual Serrin condition.

These types of results are known under the equivalence "weak = strong". In other words it is possible to show that if there exists a more regular weak solution, then the usual one

(whose existence was proved by Leray) and such a regular solution necessarily coincide. The moral of the story is that if we *postulate* more regularity on weak solutions, then the uniqueness follows. In particular this argument shows that the uniqueness, the continuous dependence on the initial data and the regularity problems for the Navier–Stokes equations are closely related. In other words, any global weak solution coincides with a more regular one as long as such a solution exists.

It is also clear from this remark and from the analysis performed in Section 3, that if a weak solution v exists and if the initial data $v_0 \in L^3$, then the solution is a strong one on some interval $[0, T)$ with $T > 0$ (hence $v(t)$ is smooth for $0 < t < T$). Moreover, we may take $T = \infty$ if $\|v_0\|_3$ is small enough. In fact, as we recalled in Section 3, there exists a strong solution $u \in \mathcal{C}([0, T); L^3)$ with $T > 0$, with $u_0 = v_0$ and satisfying Serrin's criterion. This is a simple consequence of (167) and follows directly from the result by von Wahl [232] and by Kozono and Sohr [125] (see [114]).

On the other hand, we cannot apply the uniqueness result of von Wahl to prove the uniqueness of mild solutions in $\mathcal{C}([0, T); L^p)$ (neither for the critical case $p = 3$ nor for the supercritical one $p > 3$) because the initial data only belong to L^p and, in general, not to L^2. There are of course two exceptions: the case of a bounded domain and the case of the space dimension two. As a matter of fact, if Ω_b is a bounded domain in $\mathbb{R}^3$, by means of the embedding $L^p(\Omega_b) \hookrightarrow L^2(\Omega_b)$, if $p > 2$ (rather $p \geqslant 3$ so that the existence of a solution is guaranteed, as we have seen in Section 3) and von Wahl's uniqueness theorem, it is possible to prove that Leray's weak solutions coincide with Kato's mild ones, so that their uniqueness follows in a straightforward manner [92]. In the same way, if we consider $\mathbb{R}^2$ instead of $\mathbb{R}^3$, it is obvious that the uniqueness criterion of von Wahl gives uniqueness of mild solutions with data in the critical space $L^2(\mathbb{R}^2)$ (the supercritical case $L^q(\mathbb{R}^2)$, $q \geqslant 2$, always being easier to treat as we are going to see in the following section). In other words, once again, in two dimensions there is no mystery concerning uniqueness: Leray's theory, based on the energy space $L^2(\mathbb{R}^n)$, is in a perfect agreement with Kato's one, based on the invariant space $L^n(\mathbb{R}^n)$, because the two spaces involved coincide if $n = 2$.

5.2. *Supercritical mild solutions*

From the previous discussion it is clear that we will limit ourselves to the case of the whole three-dimensional space $\mathbb{R}^3$. Of course, *mutatis mutandis*, the results of this and the following sections apply as usual to $\mathbb{R}^n$, $n \geqslant 2$, as well. A very simple case is provided by the uniqueness of mild solutions in supercritical spaces. For example, in the case of the Lebesgue spaces L^p, $p > 3$, the following result holds true:

THEOREM 11. *Let $3 < p \leqslant \infty$ be fixed. For any $v_0 \in L^p$, $\nabla \cdot v_0 = 0$, and any $T > 0$, there exists at most a mild solution in $\mathcal{C}([0, T); L^p)$ to the Navier–Stokes equations. In other words, the solution $v(t, x)$ given by Theorem 1 is unique in the space $\mathcal{C}([0, T); L^p)$.*

The proof of this property is so simple that we wish to sketch it here. Let us suppose that $v_1(t, x) \in \mathcal{C}([0, T); L^p)$ and $v_2(t, x) \in \mathcal{C}([0, T); L^p)$ solve the mild integral equation

$$v_i(t) = S(t)v_0 + B(v_i, v_i)(t), \quad i = 1, 2, \tag{221}$$

with the same initial data v_0. Then, by taking the difference between these equations

$$v_1 - v_2 = B(v_1, v_1 - v_2) + B(v_1 - v_2, v_2) \tag{222}$$

and using (138), we get

$$
\begin{aligned}
\sup_{0<t<T} & \left\| (v_1 - v_2)(t) \right\|_p \\
&\lesssim \eta(T, p) \Big(\sup_{0<t<T} \left\| v_1(t) \right\|_p + \sup_{0<t<T} \left\| v_2(t) \right\|_p \Big) \sup_{0<t<T} \left\| (v_1 - v_2)(t) \right\|_p,
\end{aligned} \tag{223}
$$

where

$$\eta(T, p) = \frac{T^{1/2(1-3/p)}}{1 - 3/p}. \tag{224}$$

We can always take $T = T'$ small enough in order to obtain

$$\eta\big(T', p\big) \Big(\sup_{0<t<T'} \left\| v_1(t) \right\|_p + \sup_{0<t<T'} \left\| v_2(t) \right\|_p \Big) < 1 \tag{225}$$

which obviously implies $v_1 = v_2$ in $\mathcal{C}([0, T'); L^p)$. Now, it is also easy to see that this argument can be iterated to get uniqueness up to time $2T'$ (and so on to $3T'$, etc.). In other words, as explained in the papers by Kato and Fujita [117] (p. 254) and [87] (p. 290), the iteration scheme is well posed and leads to uniqueness up to time T.

5.3. *Critical mild solutions*

In this section we are interested in the proof of the uniqueness of the solution given by Theorem 2. The historical details describing the achievement of this result are contained in [37] and for a systematic approach of the existence and uniqueness problem for mild solutions, the reader is also referred to the papers of Amann [1].

Let us note from the very beginning that, by a simple application of Lemma 4 and Theorem 3, it is always possible to ensure the uniqueness of a mild local solution $v(t, x)$ in a critical space (e.g., $\mathcal{C}([0, T); L^3)$) associated with an initial datum (resp. $v_0 \in L^3$, $\nabla \cdot v_0 = 0$), if we just require that it belongs to one of the auxiliary spaces described before (introduced by Weissler, Calderón and Giga) and if the norm of the solution $v(t, x)$ in such a space is smaller than a given constant (for example, smaller than $2\|v_0\|_3$, as follows directly from (119), (121) and (122)). Even if this remark is trivial and despite the fact that the condition under which the uniqueness is satisfied is very restrictive, we will use this elementary uniqueness result in Section 6 devoted to the proof of existence of self-similar solutions for the Navier–Stokes equations.

Since the introduction at the beginning of the 1960s of the mild formulation of the Navier–Stokes equations by Kato and Fujita [87,117], other results were discovered, ensuring the uniqueness of the corresponding solution under several regularity hypotheses

near $t = 0$. In the simplest case, when the solutions belong to $\mathcal{C}([0, T); L^3)$, these additional conditions are written [114] $\lim_{t \to 0} t^{\frac{\alpha}{2}} \|v(t)\|_q = 0, \alpha = 1 - \frac{3}{q}, 3 < q < \infty$, or [100], for the same values of α and q, $v \in L^{\frac{2}{\alpha}}((0, T), L^q)$. In fact, as we described in detail in Section 3.4, the use of one of these two auxiliary norms (corresponding to two auxiliary subspaces of $\mathcal{C}([0, T); L^3)$) makes it possible to apply the fixed point algorithm to obtain the existence of mild solutions in $\mathcal{C}([0, T); L^3)$. This is the reason why, in one of these subspaces, the uniqueness of the solution can be guaranteed as well. For example, the following result was known since the fundamental papers by Kato and Fujita.

THEOREM 12. *Let $3 < q \leqslant \infty$ be fixed. For any $v_0 \in L^3$, $\nabla \cdot v_0 = 0$, and any $T > 0$, there exists at most a mild solution to the Navier–Stokes equations such that $v(t, x) \in \mathcal{C}([0, T); L^3)$, $t^{1/2(1-3/q)} v(t, x) \in \mathcal{C}([0, T); L^q)$ and the following condition is satisfied*

$$\lim_{t \to 0} t^{1/2(1-3/q)} \|v(t)\|_q = 0. \tag{226}$$

In other words, using the notation of Section 3.4, Theorem 12 guarantees uniqueness (only) in the subspace $\mathcal{N} \cap \mathcal{K}_q$, $3 < q \leqslant \infty$. If $q = \infty$, the uniqueness is treated in detail in [166]. If $3 < q < \infty$, the proof follows directly from Lemma 10. In fact, if v_i, $i = 1, 2$ are two solutions that verify $t^{1/2(1-3/q)} v_i(t, x) \in \mathcal{C}([0, T); L^q)$ and $\lim_{t \to 0} t^{1/2(1-3/q)} \|v_i(t)\|_q = 0$ we have by Lemma 10 (here $3 < q < \infty$)

$$\sup_{0 < t < T} t^{1/2(1-3/q)} \|(v_1 - v_2)(t)\|_q$$

$$\lesssim \sup_{0 < t < T} t^{1/2(1-3/q)} \|(v_1 - v_2)(t)\|_q$$

$$\times \left(\sup_{0 < t < T} t^{1/2(1-3/q)} \|v_1(t)\|_q + \sup_{0 < t < T} t^{1/2(1-3/q)} \|v_2(t)\|_q \right) \tag{227}$$

and it is possible to chose $T = T'$ small enough so that

$$\left(\sup_{0 < t < T'} t^{1/2(1-3/q)} \|v_1(t)\|_q + \sup_{0 < t < T'} t^{1/2(1-3/q)} \|v_2(t)\|_q \right) < 1, \tag{228}$$

thus implying uniqueness (again, let us state that this argument can be iterated in time as in the proof of Theorem 11).

Of course, the previous result is not satisfactory and one would expect that the following result holds true.

THEOREM 13. *For any $v_0 \in L^3$, $\nabla \cdot v_0 = 0$ and any $T > 0$, there exists at most a mild solution to the Navier–Stokes equations such that $v(t, x) \in \mathcal{C}([0, T); L^3)$.*

The first proof of Theorem 13, say of the uniqueness in $\mathcal{C}([0, T); L^3)$ without any additional hypothesis (that was followed by at least five different other proofs [143,147]), was obtained in 1997 and was based on two well-known ideas. The first one is that

it is more simple to study the bilinear operator $B(v, u)(t)$ in a Besov frame [34]; the second is that it is helpful to distinguish in the solution v the contribution from the tendency $\exp(t\Delta)v_0$ and from the fluctuation $B(v, v)(t)$, the latter function always being more regular than the former [34]. More precisely, Furioli, Lemarié and Terraneo in [91,92] were able to prove the uniqueness theorem in its optimal version, say Theorem 13, by using the bicontinuity of the scalar operator $B(f, g)(t)$ (and thus the vectorial as well) respectively from $L^\infty((0, T); L^3) \times L^\infty((0, T); L^3) \to L^\infty((0, T); \dot{B}_2^{1/2,\infty})$ and from $L^\infty((0, T); \dot{B}_2^{1/2,\infty}) \times L^\infty((0, T); L^3) \to L^\infty((0, T); \dot{B}_2^{1/2,\infty})$.

What is remarkable is that, contrary to what one would expect, the spaces L^3 and $\dot{B}_2^{1/2,\infty}$ are not comparable. The fact that the Besov space of the positive regularity index played only a minor role in the paper [92] led naturally to the question whether one could do without it. Some months after the announcement of the uniqueness theorem of Lemarié and his students, Meyer showed how to improve this result. The distinction between the fluctuation and the tendency was not used, the time–frequency approach was unnecessary and the Besov spaces did not play any role. Meyer's proof shortened the problem to the bicontinuity of the bilinear term $B(f, g)(t)$ in the Lorentz space $L^{(3,\infty)}$ and more precisely, as stated in Proposition 9, in $\mathcal{C}([0, T); L^{(3,\infty)})$ [166]. This result by itself is even more surprising because, as we recalled in Section 3, Oru proved otherwise that, in spite of all the cancellations that it contains, the full vectorial bilinear term $B(v, u)(t)$ is not continuous in $\mathcal{C}([0, T); L^3)$ [183].

Let us now see how Proposition 9 simply implies Theorem 13. Let v_1 and v_2 two mild solutions in $\mathcal{C}([0, T); L^3)$ with same initial data $v_0 \in L^3$ and consider their difference

$$
\begin{aligned}
v_1 - v_2 &= B(v_1, v_1 - v_2) + B(v_1 - v_2, v_2) \\
&= B(v_1 - S(t)v_0, v_1 - v_2) + B(S(t)v_0, v_1 - v_2) \\
&\quad + B(v_1 - v_2, v_2 - S(t)v_0) + B(v_1 - v_2, S(t)v_0).
\end{aligned}
\tag{229}
$$

Now, by means of Proposition 9 (via the embedding $L^3 \hookrightarrow L^{(3,\infty)}$) and of a slight modification of Lemma 18, we get the following estimate

$$
\begin{aligned}
&\sup_{0<t<T} \left\| (v_1 - v_2)(t) \right\|_{L^{3,\infty}} \\
&\lesssim \sup_{0<t<T} \left\| (v_1 - v_2)(t) \right\|_{L^{3,\infty}} \Bigg(\sup_{0<t<T} t^{1/2(1-3/q)} \left\| S(t)v_0 \right\|_q \\
&\quad + \sup_{0<t<T} \left\| v_1 - S(t)v_0 \right\|_{L^3} + \sup_{0<t<T} \left\| v_2 - S(t)v_0 \right\|_{L^3} \Bigg),
\end{aligned}
\tag{230}
$$

where q can be chosen in the interval $3 < q \leqslant \infty$ (for instance $q = \infty$ in the proof contained in [166]). Finally, it is possible to chose $T = T'$ small enough so that

$$
\begin{aligned}
&\Bigg(\sup_{0<t<T'} t^{1/2(1-3/q)} \left\| S(t)v_0 \right\|_q \\
&\quad + \sup_{0<t<T'} \left\| v_1 - S(t)v_0 \right\|_{L^3} + \sup_{0<t<T'} \left\| v_2 - S(t)v_0 \right\|_{L^3} \Bigg) < 1,
\end{aligned}
\tag{231}
$$

this property being a direct consequence of Lemma 9 and of the strong continuity in time of the L^3 norm of the solutions v_1 and v_2. From this estimate we deduce that locally in time $v_1 - v_2$ is equal to zero in the sense of distribution, thus $v_1 - v_2$ is equal to zero in L^3 in the interval $0 \leqslant t \leqslant T'$ and the argument can of course be iterated in the time variable.

The proof of the uniqueness of the solution in the more general cases given by Theorems 3–5 (say, when the initial data belongs to a Besov space) is contained in [92].

To conclude, we wish to present a different proof of the uniqueness result from the one contained in [166], based on Proposition 9. In fact, following [36,48], we will give here a more precise result.

PROPOSITION 10. *Let* $3/2 < q < \infty$ *and* $0 < T \leqslant \infty$ *be fixed. The bilinear operator* $B(f, g)(t)$ *is bicontinuous from* $L^\infty((0, T); L^{(3,\infty)}) \times L^\infty((0, T); L^{(3,\infty)}) \to L^\infty((0, T); \dot{B}_q^{3/q-1,\infty})$.

We will prove this proposition by duality, as we did in the proof of Lemmas 20 and 21. Let us consider a test function $\chi(x) \in C_0^\infty$ and evaluate the duality product in $\mathbb{R}^3$ with the bilinear term. We get

$$\left| \langle B(f, g)(t), \chi \rangle \right| \leqslant \int_0^t \left| \left\langle s^{-2} \Theta\left(\frac{\cdot}{\sqrt{s}} \right) * \chi, (fg)(t-s) \right\rangle \right| ds. \tag{232}$$

If we had at our disposal a generalization of the classical Young's inequality

$$\|a * b\|_\infty \leqslant \|a\|_{3/2} \|b\|_3, \tag{233}$$

we could hope to modify the following argument that gives the continuity of $B(f, g)$ from $L^\infty((0, T); L^3) \times L^\infty((0, T); L^3) \to L^\infty((0, T); \dot{B}_{3/2}^{1,\infty})$, that is,

$$\begin{aligned}
\left| \langle B(f, g)(t), \chi \rangle \right| & \\
&\leqslant \left(\sup_{0 < t < T} \|fg(t)\|_{3/2} \right) \int_0^t \left\| s^{-2} \Theta\left(\frac{\cdot}{\sqrt{s}} \right) * \chi \right\|_3 ds \\
&\leqslant 2 \left(\sup_{0 < t < T} \|f(t)\|_3 \right) \left(\sup_{0 < t < T} \|g(t)\|_3 \right) \int_0^\infty u \left\| \frac{1}{u^3} \Theta\left(\frac{\cdot}{u} \right) * \chi \right\|_3 \frac{du}{u} \\
&\lesssim \left(\sup_{0 < t < T} \|f(t)\|_3 \right) \left(\sup_{0 < t < T} \|g(t)\|_3 \right) \|\chi\|_{\dot{B}_3^{-1,1}},
\end{aligned} \tag{234}$$

the last estimate being a consequence of the equivalence of Besov norms given in Proposition 3.

Now, the generalized Young's inequality applied to the Lorentz spaces [111]

$$\|a * b\|_r \leqslant C_{p,q} \|f\|_p \|g\|_{(q,\infty)} \tag{235}$$

holds only if $1 < p, q, r < \infty$ and $p^{-1} + q^{-1} = 1 + r^{-1}$. Thus, there is no hope of modifying (233).

To circumvent such a difficulty, we will decompose the kernel Θ in two parties Θ_1 and Θ_2 defined by their Fourier transforms as

$$\widehat{\Theta}_1(\xi) =: |\xi| e^{-|\xi|^2/2} \tag{236}$$

and

$$\widehat{\Theta}_2(\xi) =: e^{-|\xi|^2/2}, \tag{237}$$

in such a way that

$$|\xi| \exp\big[-s|\xi|^2\big] = \frac{1}{\sqrt{s}} \widehat{\Theta}\big(\sqrt{s}\xi\big) = \frac{1}{\sqrt{s}} \widehat{\Theta}_1\big(\sqrt{s}\xi\big) \widehat{\Theta}_2\big(\sqrt{s}\xi\big). \tag{238}$$

With this decomposition, we can write, by taking the inverse Fourier transform (p and q being conjugate exponents)

$$\big|\langle B(f,g)(t), \chi \rangle\big|$$

$$\leqslant \int_0^t \left|\left\langle s^{-2}\Theta_1\left(\frac{\cdot}{\sqrt{s}}\right) * \chi, \left(\frac{1}{\sqrt{s}}\right)^3 \Theta_2\left(\frac{\cdot}{\sqrt{s}}\right) * fg(t-s) \right\rangle\right| ds$$

$$\leqslant \int_0^t \left\|\left(\frac{1}{\sqrt{s}}\right)^3 \Theta_2\left(\frac{\cdot}{\sqrt{s}}\right) * fg(t-s)\right\|_q \left\|s^{-2}\Theta_1\left(\frac{\cdot}{\sqrt{s}}\right) * \chi\right\|_p ds \tag{239}$$

and Young's generalized inequality ($3/2 < q < \infty$, $q^{-1} + 1 = \alpha^{-1} + 2/3$)

$$\left\|\left(\frac{1}{\sqrt{s}}\right)^3 \Theta_2\left(\frac{\cdot}{\sqrt{s}}\right) * fg(t-s)\right\|_q \lesssim \left\|\left(\frac{1}{\sqrt{s}}\right)^3 \Theta_2\left(\frac{\cdot}{\sqrt{s}}\right)\right\|_\alpha \|fg(t-s)\|_{(3/2,\infty)}$$

$$\lesssim s^{-3/2(2/3-1/q)} \|fg(t-s)\|_{(3/2,\infty)} \tag{240}$$

allows to conclude

$$\big|\langle B(f,g)(t), \chi \rangle\big|$$

$$\leqslant \left(\sup_{0<t<T} \|fg(t)\|_{(3/2,\infty)}\right) \int_0^t \frac{\|s^{-2}\Theta_1(\frac{\cdot}{\sqrt{s}}) * \chi\|_p}{s^{\frac{3}{2}(2/3-1/q)}} ds$$

$$\leqslant 2\left(\sup_{0<t<T} \|f(t)\|_{(3,\infty)}\right)\left(\sup_{0<t<T} \|g(t)\|_{(3,\infty)}\right) \int_0^\infty \frac{\|\frac{1}{u^3}\Theta_1(\frac{\cdot}{u}) * \chi\|_p}{u^{1-3/q}} \frac{du}{u}$$

$$\lesssim \left(\sup_{0<t<T} \|f(t)\|_{(3,\infty)}\right)\left(\sup_{0<t<T} \|g(t)\|_{(3,\infty)}\right) \|\chi\|_{\dot{B}_p^{1-3/q,1}}. \tag{241}$$

In order to make use of Proposition 10 in the proof of Theorem 13 we need a classical result (see [10]).

LEMMA 24. *The following embedding are continuous*: $\dot{B}_q^{3/q-1,\infty} \hookrightarrow L^{(3,\infty)}$ *for any* $0 < q < 3$ *and* $L^{(3,\infty)} \hookrightarrow \dot{B}_q^{3/q-1,\infty}$ *for any* $3 < q < \infty$.

Without losing generality, let us prove this lemma only when $q = 2$. In order to do this, we make use of the characterization of Besov and Lorentz spaces given by the interpolation theory as stated in (115) (see [10])

$$\left(L^2, L^4\right)_{(2/3,\infty)} = L^{(3,\infty)} \tag{242}$$

and

$$\left(\dot{B}_2^{0,1}, \dot{B}_2^{3/4,1}\right)_{(2/3,\infty)} = \dot{B}_2^{1/2,\infty}. \tag{243}$$

Now, as

$$\dot{B}_2^{0,1} \hookrightarrow L^2 \tag{244}$$

and

$$\dot{B}_2^{3/4,1} \hookrightarrow \dot{B}_4^{0,1} \hookrightarrow L^4, \tag{245}$$

we get the required result

$$\dot{B}_2^{1/2,\infty} \hookrightarrow L^{(3,\infty)}. \tag{246}$$

Proposition 9 is proved and Theorem 13 follows (see [166]).

6. Self-similar solutions

The viscous flows for which the profiles of the velocity field at different times are invariant under a scaling of variables are called self-similar. More precisely, we are talking about solutions to the Navier–Stokes equations

$$\begin{aligned}
&\frac{\partial v}{\partial t} - \nu \Delta v = -(v \cdot \nabla)v - \nabla p, \\
&\nabla \cdot v = 0, \\
&v(0) = v_0
\end{aligned} \tag{247}$$

such that

$$v(t,x) = \lambda(t)V\left(\lambda(t)x\right), \qquad p(t,x) = \lambda^2(t)P\left(\lambda(t)x\right), \tag{248}$$

$\lambda(t)$ being a function of time, $P(x)$ a function of x and $V(x)$ a divergence-free vector field.

Two possibilities arise in what follows.

DEFINITION 10 (Backward). A backward self-similar solution is a solution of the form (248), where $\lambda(t) = 1/\sqrt{2a(T-t)}$, $a > 0$, $T > 0$ and $t < T$. As such, $V(x)$ and $P(x)$ solve the system

$$-\nu\Delta V + aV + a(x \cdot \nabla)V + (V \cdot \nabla)V + \nabla P = 0,$$
$$\nabla \cdot V = 0. \tag{249}$$

DEFINITION 11 (Forward). A forward self-similar solution is a solution of the form (248), where $\lambda(t) = 1/\sqrt{2a(T+t)}$, $a > 0$, $T > 0$ and $t > -T$. As such, $V(x)$ and $P(x)$ solve the system

$$-\nu\Delta V - aV - a(x \cdot \nabla)V + (V \cdot \nabla)V + \nabla P = 0,$$
$$\nabla \cdot V = 0. \tag{250}$$

6.1. *Backward: Singular*

The motivation for studying backward self-similar solutions is that, if they exist, they would possess a singularity when $t = T$; indeed $\lim_{t \nearrow T} \|\nabla v(t)\|_2 = \infty$. In 1933, Leray remarked that if a weak solution v becomes "turbulent" at a time T, then the quantity $u(t) = \sup_{x \in \mathbb{R}^3} \sqrt{v \cdot v}$ has to blow-up like $\frac{1}{\sqrt{2a(T-t)}}$ when t tends to T. Furthermore, he suggested, without proving their existence, to look for backward self-similar solutions. His conclusion was the following [148]:

> [...] unfortunately I was not able to give an example of such a singularity [...]. If I had succeeded in constructing a solution to the Navier equations that becomes irregular, I would have the right to claim that turbulent solutions not simply reducing to regular ones do exist. But if this position were wrong, the notion of turbulent solution, that for the study of viscous fluids will not play a key role any more, would not lose interest: there have to exist some problems of Mathematical Physics such that the physical causes of regularity are not sufficient to justify the hypothesis introduced when the equations are derived; to these problems we can apply similar considerations of the ones advocated so far.

The first proof of the nonexistence of backward self-similar solutions sufficiently decreasing at infinity seems to have been given by a physicist at the beginning of the 1970s in a somewhat esoteric paper, written by Rosen [203]. Another argument for the nonexistence of nontrivial solutions to the system (249) was given by Foias and Temam in [81].

But the mathematical proof for the nonexistence of backward self-similar solutions as imagined by Leray was available in functional spaces only later, in 1996, thanks to the works of the Czech school of J. Nečas.

In a paper published in the French Academy "Comptes Rendus" [179] – the last one to be presented by Leray (1906–1998) – Nečas, Růžička and Šverák announced that any weak solution V to the Navier–Stokes equations (249) belonging to the space $L^3 \cap W^{1,2}_{\text{loc}}$ reduces to the zero solution. The proof of this remarkable statement [180] is based on asymptotic estimates at infinity (in the Caffarelli–Kohn–Nirenberg sense) for the functions

V and P as well as for their derivatives, and on the maximum principle for the function $\Pi(x) = \frac{1}{2}|V(x)|^2 + P(x) + ax \cdot V(x)$ on a bounded domain of $\mathbb{R}^3$. A different approach to obtain the same result, without using the Caffarelli–Kohn–Nirenberg theory, but under the more restrictive condition $V \in W^{1,2}$ was proposed afterwards by Málek, Nečas, Pokorný and Schonbek [155] (see also [170] for a generalization of the method to the proof of nonexistence of pseudo self-similar solutions).

Now, if we impose that the norms of v that appear naturally in the energy equality derived from (247) are finite, we get the estimates $\int_{\mathbb{R}^3} |V|^2 < \infty$ and $\int_{\mathbb{R}^3} |\nabla V|^2 < \infty$, i.e., $V \in W^{1,2}$ which implies $V \in L^3$, by Sobolev embedding. But if, on the contrary, we only impose that the local version of the energy equality is finite, in other words $V \in W^{1,2}_{\text{loc}}$, we get some conditions that do not imply $V \in L^3$. This case, left open in [155,180], was solved by Tsai and gave origin to the following theorem [227,146]:

THEOREM 14. *Any weak backward self-similar solution V to the Navier–Stokes equations* (249) *belonging either to the space L^q, $3 < q < \infty$ or to $W^{1,2}_{\text{loc}}$ reduces to the zero solution.*

6.2. *Forward: Regular or singular*

As we will see in this section, the situation is more favorable in the case of mild forward self-similar solutions. In fact, since the pioneering paper of Giga and Miyakawa [107], we know of the existence of many mild forward self-similar solutions of the type (248) with $\lambda(t) = 1/\sqrt{t}$. These solutions cannot be of finite energy. In fact, if we consider the inner product between V and the equation (250) and integrate by parts in the whole space, we get, if V is sufficiently decreasing at infinity

$$\int_{\mathbb{R}^3} |\nabla V|^2 + a \int_{\mathbb{R}^3} |V|^2 = 0. \tag{251}$$

Finally, this equality results in the conclusion that $V = 0$, in particular when $V \in W^{1,2}$. (It is important to stress here that such a conclusion is not true for backward self-similar solutions because of the difference of signs in (249) and (250).)

This is why Giga and Miyakawa suggested, as an alternative to Sobolev spaces, to consider the Morrey–Campanato ones. They succeeded in proving the existence and the uniqueness of mild forward self-similar solutions to the Navier–Stokes equations written in terms of the vorticity as unknown, without applying their method to the Navier–Stokes equations in terms of the velocity. Four years later, Federbush [78,79] considered the super-critical Morrey–Campanato spaces $\dot{M}^q_2$, $3 < q < \infty$, for these equations. The critical space $\dot{M}^3_2$ was treated shortly after by Taylor [216] who, surprisingly, did not take advantage of this space which contains homogeneous functions of degree -1, to get the existence of self-similar solutions as shown in [34].

As pointed out in the previous section, a remarkable property of the Besov spaces is that they contain homogeneous functions of degree -1 among their elements, such as,

e.g., $|x|^{-1}$. This is a crucial point if we look for solutions to the Navier–Stokes equations which satisfy the scaling property

$$v(t, x) = v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x) \quad \forall \lambda > 0 \tag{252}$$

or, equivalently, taking $\lambda^2 t = 1$, such that

$$v(t, x) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right). \tag{253}$$

In fact, whenever they exist, these particular ($a = 1/2$ and $T = 0$) forward self-similar solutions $v(t, x)$ are such that their initial value $v(0, x)$ is a homogeneous function of degree -1.

We will show here how to obtain, by using a generalization of Kato's celebrated Theorem 3, the existence of mild forward self-similar solutions $v(t, x)$ with initial data v_0 homogeneous of degree -1, divergence-free and sufficiently small in a Besov space. In [34,35,44,45], we showed how to construct mild forward self-similar solutions for the Navier–Stokes equations (247), by using Besov spaces. In particular, the existence of regular forward self-similar solutions of the form $\frac{1}{\sqrt{t}} V(\frac{x}{\sqrt{t}})$ with $V \in L^q$ and $3 < q < \infty$ is contained as a corollary in [34]. The main idea of the aforementioned papers is to study the Navier–Stokes equations by the fixed point algorithm in a critical space containing homogeneous functions of degree -1. Furthermore, as noted by Planchon [186], the equivalence between the integral mild equation and the elliptic problem (250) is completely justified.

The result we are talking about is the following theorem.

THEOREM 15. *Let $3 < q < \infty$, and $\alpha = 1 - 3/q$ be fixed. There exists a constant $\delta_q > 0$ such that for any initial data $v_0 \in \dot{B}_q^{-\alpha,\infty}$, homogeneous of degree -1, $\nabla \cdot v_0 = 0$ in the sense of distributions and such that*

$$\|v_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta_q, \tag{254}$$

then there exists a global mild forward self-similar solution $v(t, x)$ to the Navier–Stokes equations such that

$$v(t, x) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right) \tag{255}$$

where $V(x)$ is a divergence-free function belonging to $\dot{B}_q^{-\alpha,\infty} \cap L_q$.

The proof of these results follows by a simple modification of Theorem 3, once we recall that it is always possible to ensure the uniqueness of a mild solution $v(t, x)$ in a critical space, if the norm of the solution $v(t, x)$ in such a space is smaller than a given constant (see Section 5.3). In fact, suppose that $v(t, x)$ solves Navier–Stokes with a datum $v_0 \in \dot{B}_q^{-\alpha,\infty}$

such that $v_0 = \lambda v_0(\lambda x)$ $\forall \lambda > 0$, then the corresponding solution $v(t, x)$, whose uniqueness is ensured if $\sup_{0 < t < \infty} t^{\alpha/2} \|v(t, x)\|_q \leqslant C$, has to coincide with $\lambda v(\lambda^2 t, \lambda x)$ $\forall \lambda > 0$ for the latter inequality is invariant under the same self-similar scaling.

Since 1995, Barraza has suggested replacing the Besov spaces with the Lorentz ones $L^{(3,\infty)}$ (see also Kozono and Yamazaki's results [131,133,236]), always with the aim of proving the existence of forward self-similar solutions [4], but he did not achieve the bicontinuity of the bilinear operator in this space. This result was proven later by Meyer (see Proposition 9), and was applied, not only to obtain the uniqueness of Kato's mild solutions (Theorem 13), but also to prove the existence of forward self-similar solutions. More precisely:

THEOREM 16. *There exists a constant $\delta > 0$ such that for any initial data $v_0 \in L^{(3,\infty)}$, homogeneous of degree -1, $\nabla \cdot v_0 = 0$ in the sense of distributions and such that*

$$\|v_0\|_{L^{(3,\infty)}} < \delta, \tag{256}$$

then there exists a global mild forward self-similar solution $v(t, x)$ to the Navier–Stokes equations such that

$$v(t, x) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right), \tag{257}$$

where $V(x)$ is a divergence-free function belonging to $L^{(3,\infty)}$.

Once again, the proof of this theorem is trivial if we recall the bicontinuity of the bilinear term $B(f, g)(t)$ in $\mathcal{C}([0, T); L^{(3,\infty)})$ [166] (see Proposition 9). This result shows that there is no need for Fourier transform or Besov spaces to prove the existence of self-similar solutions for Navier–Stokes.

As we have already pointed out, Le Jan and Sznitman [137,138] gave an even simpler ad hoc setting to prove such a result. The space they introduced is defined, however, by means of a Fourier transform condition. More exactly, following the notations of Section 2.5.3,

$$\psi \in \mathcal{PM}^2 \quad \text{if and only if} \quad \hat{\psi} \in L^1_{\text{loc}} \text{ and } \|\psi\|_{\mathcal{PM}^2} = \sup_{\xi} |\xi|^2 |\hat{\psi}(\xi)| < \infty. \tag{258}$$

Now, according to the simplified version of Le Jan and Sznitman's result contained in [48], we have:

THEOREM 17. *The bilinear operator $B(f, g)$ is bicontinuous from $L^\infty_t(\mathcal{PM}^2) \times L^\infty_t(\mathcal{PM}^2)$ into $L^\infty_t(\mathcal{PM}^2)$. Therefore there exists a unique global mild solution to the Navier–Stokes equations in $L^\infty_t(\mathcal{PM}^2)$ provided the initial data is divergence-free and sufficiently small in $\mathcal{PM}^2$.*

Note that the authors made use of some probabilistic tools in [137,138] requiring rather subtle techniques to obtain the continuity of the bilinear operator. More precisely, the main

idea contained in these papers is to study the non linear integral equation verified by the Fourier transform of the Laplacian of the velocity vector field associated with the "deterministic equations" of Navier–Stokes. This integral representation involves a Markovian kernel K_ξ, associated to the branching process, called stochastic cascades, in which each particle located at $\xi \neq 0$, after an exponential holding time of parameter $|\xi|^2$, with equal probability either dies out or gives birth to two descendants, distributed according to K_ξ. By taking the inverse Fourier transform one can thus obtain a solution to the Navier–Stokes equations . . . arising from a sequence of cascades!

However, as pointed out in [48], in the particular case of the pseudo-measures, Theorem 17 is a straightforward consequence of the fixed point algorithm and it is enough to show why the bilinear operator is bicontinuous. We work in Fourier space, with $\hat{f}$ and $\hat{g}$ instead of f and g. A standard argument (rotational invariance and homogeneity) shows that [214,215]

$$\frac{1}{|\xi|^2} * \frac{1}{|\xi|^2} \simeq \frac{C}{|\xi|}. \tag{259}$$

Thus

$$\widehat{B(f,g)}(t,\xi) = \int_0^t |\xi| \mathrm{e}^{-(t-s)|\xi|^2} \hat{f}(s) * \hat{g}(s)\, \mathrm{d}s, \tag{260}$$

and, upon using (259),

$$\sup_{t,\xi}\big(|\xi|^2\big|\widehat{B}(t)\big|\big) \lesssim \sup_{t,\xi}\big(|\xi|^2\big|\hat{f}(t)\big|\big) \sup_{t,\xi}\big(|\xi|^2\big|\hat{g}(t)\big|\big) \sup_{t,\xi} \int_0^t |\xi|^2 \mathrm{e}^{-(t-s)|\xi|^2}\, \mathrm{d}s. \tag{261}$$

This last integral is in turn less than unity, which concludes the proof once the fixed point algorithm is recalled.

Finally, the norm of the space $\mathcal{PM}^2$ being critical in the sense of Definition 9, the following result can be easily deduced from the previous estimate.

THEOREM 18. *There exists a constant $\delta > 0$ such that for any initial data $v_0 \in \mathcal{PM}^2$, homogeneous of degree -1, $\nabla \cdot v_0 = 0$ in the sense of distributions and such that*

$$\|v_0\|_{\mathcal{PM}^2} < \delta, \tag{262}$$

then there exists a global mild forward self-similar solution $v(t,x)$ to the Navier–Stokes equations such that

$$v(t,x) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right), \tag{263}$$

where $V(x)$ is a divergence-free function belonging to $\mathcal{PM}^2$.

REMARK. As far as backward self-similar solutions are concerned, we can exclude the existence of singularities for the Navier–Stokes equations simply by using Nečas, Růžička and Šverák and Tsai's results. However, singular forward self-similar solutions may exist. More precisely, there is a substantial difference between the self-similar solutions constructed in Theorem 15 and those constructed in Theorems 16 and 18. Both have a singularity at time $t = 0$ (of the type $\sim 1/|x|$), but the solution constructed in Theorem 15 becomes *instantaneously* smooth for $t > 0$, whereas this property cannot be ensured a priori for the other two families of self-similar solutions. The reason is the following. Even if they are both issued from the fixed point algorithm, the solutions in Theorem 15, and in Theorems 16 and 18 are constructed in a very different way. In the first case, in order to overcome the difficulty (and sometimes the impossibility) of proving the continuity of the bilinear estimate in the so-called critical spaces, we had to make use of Kato's celebrated idea of considering *two norms* at the same time, the so-called natural norm and the auxiliary regularizing norm. As such, Kato's approach imposes a priori a regularization effect on the solutions we look for. In other words, they are considered as fluctuations around the solution of the heat equation with the same initial data. In the case of the self-similar solution arising from Theorem 15, this regularity condition is imposed by the Lebesgue norm. More explicitly, not only does the divergence-free function $V(x)$ belong to the Besov space $\dot{B}_q^{-\alpha,\infty}$, but also to L_q, which is not a priori the case for the solutions in Theorems 16 and 18.

For people who believe in blow-up and singularities, this a priori condition coming from the *two norms* approach is indeed very strong. In other words, at variance with Leray's approach, Kato's algorithm does not seem to provide a framework for studying a priori singular solutions. However, as we have seen in the previous pages, two exceptions exist, i.e., two critical spaces where Kato's method applies with just *one norm*: the Lorentz space $L^{(3,\infty)}$ (considered independently by Kozono and Yamazaki [131,133,236], Barraza [4,5], Meyer [166]) and the pseudo-measure space of Le Jan and Sznitman [137,138]. The approach with only *one norm* gives the existence of a solution in a larger space which, in our case, contains genuinely singular solutions that are not smoothened by the action of the nonlinear semigroup associated.

The importance of this remark will be clear in Section 6.4, where we will construct explicit forward self-similar solutions, singular for any time $t \geqslant 0$, and we will suggest how to obtain loss of smoothness for solutions with large data.

If the debate concerning singularities is still open, as far as Besov spaces and harmonic analysis tools are concerned, it is clear that they have nothing to do with the existence (Theorem 16) nor the nonexistence (Theorem 14) of self-similar solutions.

6.3. *Asymptotic behavior*

Finding self-similar solutions is important because of their possible connection with attractor sets. In other words, they are related to the asymptotic behavior of global solutions of the Navier–Stokes equations. A heuristic argument is the following: let $v(t, x)$ be a global solution to the Navier–Stokes system, then, for any $\lambda > 0$, the function

$v_\lambda(t, x) =: \lambda v(\lambda^2 t, \lambda x)$ is also a solution to the same system. Now, if in a "certain sense" the limit $\lim_{\lambda \to \infty} v_\lambda(t, x) =: u(t, x)$ exists, then it is easy to see that $u(t, x)$ is a self-similar solution and that $\lim_{t \to \infty} \sqrt{t} v(t, \sqrt{t} x) = u(1, x)$. In [186,188,189], Planchon gave the precise mathematical frame to explain the previous heuristic argument (see also Meyer [164], Barraza [5] and, for more general nonlinear equations, Karch [113]).

As we suggested among the open problems in [34], the existence of self-similar solutions also evokes the study of exact solutions for Navier–Stokes. In the following section, we will describe the result of Tian and Xin, who gave an explicit one-parameter family of self-similar solutions, singular in a single point [221], and we will show how to interpret their result as a loss of smoothness for large data.

We would like to mention here the papers of Okamoto [181,182] that contain a systematic study of exact solutions of the systems (249) and (250). These results merit attention, especially since the resolution of these elliptic equations seems very difficult. One could image to apply these results to the study of mild solutions in the subcritical case, for which neither the existence nor the uniqueness is known (see also [37]) unless some restriction are required (see [32,33,145]).

More precisely, let us suppose that we can prove the existence of a nontrivial self-similar solution $v(t, x) = \frac{1}{\sqrt{t}} V(\frac{x}{\sqrt{t}})$ – in other words a solution V of (250) – with $V \in L^p$ and $1 \leqslant p < 3$. Then the Cauchy problem associated to the zero initial data would allow two different solutions, viz. v and 0, both belonging to $\mathcal{C}([0, T]; L^p)$. In fact, $\lim_{t \to 0} \|\frac{1}{\sqrt{t}} V(\frac{x}{\sqrt{t}})\|_p = 0$, provided $1 \leqslant p < 3$. And the Cauchy problem would be ill-posed in $\mathcal{C}([0, T]; L^p)$, $1 \leqslant p < 3$ in the same way that it is ill posed for a semilinear partial differential equation studied in 1985 by Haraux and Weissler [108].

This point of view should confirm the conjecture formulated by Kato [116], according to which the Cauchy problem is ill posed in the sense of Hadamard when $1 \leqslant p < 3$. In the case $p = 2$, for example, we will not obtain a unique, global, regular and stable solution and the scenario imagined by Leray would be possible. We will come back to this question in Section 7.2.

Finally, let us quote the book of Giga and Giga [102] "*Nonlinear Partial Differential Equations – Asymptotic Behavior of Solutions and Self-Similar Solutions*", whose English translation should be available soon, that contains one of the most comprehensive and self-contained state of the art of the results available in this direction for the Navier–Stokes and other partial differential equations (e.g., the porous medium, the nonlinear Schrödinger and the KdV equations).

6.4. *Loss of smoothness for large data?*

As we recalled in the Introduction, a question intimately related to the uniqueness problem is the regularity of the solutions to the Navier–Stokes equations. Several possibilities can be conjectured. One may imagine that blow-up of initially regular solutions never happens, or that it becomes more likely as the initial norm increases, or that there is blow-up, but only on a very thin set of probability zero. Or it is "possible that singular solutions exist but are unstable and therefore difficult to construct analytically and impossible to detect numer-

ically [. . .], which would contradict the almost universal assumption that these equations
are globally regular"[122].

As we have seen in Section 3, when using a fixed point approach, existence and unique-
ness of global solutions are guaranteed only under restrictive assumptions on the initial
data, that is required to be small in some sense, i.e., in some functional space. In Section 4
we pointed out that fast oscillations are sufficient to make the fixed point scheme work,
even if the norm in the corresponding function space of the initial data is arbitrarily large
(in fact, a different auxiliary norm turns out to be small). Here we would like to suggest
how some particular data, arbitrarily large (not oscillating) could give rise to singular so-
lutions. It is extremely unpleasant that we have no criteria to decide whether for arbitrarily
large data the corresponding solution is regular or singular.

As observed by Heywood in [110], in principle "it is easy to construct a singular solution
of the NS equations that is driven by a singular force. One simply constructs a solenoidal
vector field *u* that begins smoothly and evolves to develop a singularity, and then defines
the force to be the residual".

Recently, Tian and Xin [221] found explicit formulas for a one-parameter family of
stationary "solutions" of the three-dimensional Navier–Stokes system (1) "with $\phi \equiv 0$"
which are regular except at a given point. These explicit "solutions" agree with those pre-
viously obtained by Landau for special values of the parameter (see [135,136]). Due to the
translation invariance of the Navier–Stokes system, one can assume that the singular point
corresponds to the origin. More exactly, the main theorem from [221] reads as follows.

*All solutions to the Navier–Stokes system (with $\phi \equiv 0$) $u(x) = (u_1(x), u_2(x), u_3(x))$ and
$p = p(x)$ which are steady, symmetric about x_1-axis, homogeneous of degree -1, regular
except $(0, 0, 0)$ are given by the following explicit formula*:

$$
\begin{aligned}
u_1(x) &= 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \\
u_2(x) &= 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\
u_3(x) &= 2\frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\
p(x) &= 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2},
\end{aligned}
\tag{264}
$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and c is an arbitrary constant such that $|c| > 1$.

It is clear that these stationary "solutions" are self-similar, because they do not depend
on time and they are homogeneous of degree -1 in the space variable. Moreover, there is
no hope of describing the "solutions" given by (264) in Leray's theory, because they are
not globally of finite energy; in other words, they do not belong to L^2. However, they do
belong to L^2_{loc} and this is at least enough to allow us to give a (distributional) meaning to the

nonlinear term $(v \cdot \nabla)v = \nabla \cdot (v \otimes v)$. Finally, as pointed out at the end of Section 6.2, the "solutions" discovered by Tian and Xin cannot be analyzed by Kato's two norms method either, because they are global but not smooth. More precisely, they are singular at the origin with a singularity of the kind $\sim 1/|x|$ for all time.

There are at least two ad hoc frameworks for studying such singularity within the fixed point scheme and without using the two norms approach. We are thinking of the Lorentz space $L^{(3,\infty)}$ ([42]) and the pseudo-measure space $\mathcal{PM}^2$ ([40]), because they both contain singularities of the type $\sim 1/|x|$. However, the latter space has the advantage that not only the definition of its norm is very elementary and simplifies the calculations, it will also allow us to treat singular (Delta type) external force, that precisely arise from Tian and Xin's "solutions".

More exactly, by straightforward calculations performed in [40], one can check that, indeed, $(u_1(x), u_2(x), u_3(x))$ and $p(x)$ given by (264) satisfy the Navier–Stokes equations with $\phi \equiv 0$ in the *pointwise sense* for every $x \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. On the other hand, if one treats $(u(x), p(x))$ as a *distributional or generalized* solution to the Navier–Stokes equations in the whole $\mathbb{R}^3$, they correspond to the very singular external force $\phi = (b\delta_0, 0, 0)$, where δ_0 stands for the Dirac delta and the parameter b depends on c and $\lim_{|c| \to \infty} b(c) = 0$. As such, if c is small enough, the existence of these solutions can be ensured as well via the fixed point algorithm as in [42,40].

The stationary solutions defined in (264) are singular with singularity of the kind $\mathcal{O}(1/|x|)$ as $|x| \to 0$. This is a critical singularity, because as it was shown by Choe and Kim [65], every pointwise stationary solution to the Navier–Stokes system with $F \equiv 0$ in $B_R \setminus \{0\} = \{x \in \mathbb{R}^3 : 0 < |x| < R\}$ satisfying $u(x) = o(1/|x|)$ as $|x| \to 0$ is also a solution in the sense of distributions in the whole B_R. Moreover, it is shown in [65] that under the additional assumption $u \in L^q(B_R)$ for some $q > 3$, then the stationary solution $u(x)$ is smooth in the whole ball B_R. In other words, if $u(x) = o(1/|x|)$ as $|x| \to 0$ and $u \in L^q(B_R)$ for some $q > 3$, then the singularity at the origin is removable.

We are now ready to state our remark about a possible loss of regularity of solutions with large data (see [40]).

REMARK. Let us consider the Navier–Stokes equations with external force $\phi \equiv 0$. Then, if one defines the functions $u_\varepsilon(x, 0) = \varepsilon u(x)$, where $u(x)$ is the (divergence-free, homogeneous of degree -1) function given by (264) as the initial data, then for small ε the system has a global regular (self-similar) solution which is even more regular than a priori expected and for $\varepsilon = 1$ the system has a singular "solution" for any time. The fact that, for small ε and external force $\phi \equiv 0$ for every $x \in \mathbb{R}^3$, the solution is smooth follows from a parabolic regularization effect analyzed in [40]. On the other hand, if $\varepsilon > 1$ nothing can be said in general and the corresponding solution can be regular or singular.

However, after a more careful analysis, one realizes that this possible loss of smoothness result does not apply in the "distributional" sense, but, as we explained before, only "pointwise" for every $x \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. On the other hand, as explained in a forthcoming paper [18], this loss of smoothness for large data holds in the distributional sense for a model equation of gravitating particles (for which, moreover, blow-up is known).

7. Stability

As we have seen in the previous sections, when using a fixed point approach, existence and uniqueness of global solutions are guaranteed only under restrictive assumptions on the initial data, that is required to be small in some sense, i.e., in some functional space. In Section 4 we pointed out that fast oscillations are sufficient to make the fixed point scheme work, even if the norm in the corresponding function space of the initial data is arbitrarily large (in fact, a different auxiliary norm turns out to be small). On the other hand, in Section 6 we suggested how arbitrarily large data (not oscillating) could give rise to irregular solutions: in general, we do not know whether for arbitrarily large data the corresponding solution is regular or singular.

For the Navier–Stokes equations one might consider the entire question irrelevant, for the solution is unique and regular for small initial data and no viscous flow can be considered incompressible if the initial data are too large. The problem here is different: the set ($\delta > 0$) of initial data for which one can ensure the existence and the uniqueness ($\|v_0\| < \delta$) is not known precisely and could be too small, and the result meaningless from a physical point of view. In other words, the initial data as well as the unique corresponding solution would be "physically" zero! The "physical" role played by the smallness assumption on the initial data will be dealt with in this section. More precisely, we will make the link between this property, the stability of the corresponding global solution and the existence of Lyapunov functions.

First of all, let us note that the smallness condition is not absolute, but relative to the viscosity ν and, if we do not rescale the variables as we did in Section 3.2, this condition is written $\|v_0\|/\nu < \delta$. Now, if we interpret $\|v_0\|$ as the characteristic velocity of the problem and we suppose (in the whole space $\mathbb{R}^3$ or $\mathbb{T}^3$) the characteristic length is normalized to unity, then the quotient $R =: \|v_0\|/\nu$ can be interpreted as a Reynolds number associated with the problem. More precisely, the complexity of the Navier–Stokes equations is essentially due to the competition between the nonlinear convection term $\rho(v \cdot \nabla)v$, and the linear term of viscous diffusion, $\mu \Delta v$. The order of magnitude of the quotient between these terms (dimension equation)

$$\frac{|\rho(v \cdot \nabla)v|}{|\mu \Delta v|} \equiv \frac{\rho}{\mu} \frac{V^2/L}{V/L^2} = \frac{LV}{\nu} =: R \tag{265}$$

defines a dimensionless quantity R, called Reynolds number, that allows a comparison of the inertial forces and the viscosity ones.

Thus, the condition giving the existence and uniqueness of Kato's (global and regular) solution is nothing but by the smallness of a dimensionless Reynolds number associated with the problem. At this point it would be tempting to prove that for Reynolds numbers that are too large, the solution does not exist, or is not regular, or not unique or simply not stable. This point of view would be confirmed by the image of developed turbulence formulated in 1944 by Landau [136]:

> Yet not every solution of the equations of motion, even if exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable. For the flow to be stable it is necessary that small perturbations, if they arise, should decrease with

time. If, on the contrary, the small perturbations which inevitably occur in the flow tend to increase with time, then the flow is absolutely unstable. Such a flow unstable with respect to infinitely small perturbations cannot exist.

The criteria to find the critical Reynolds numbers above which solutions of the Navier–Stokes could not necessarily be stable under small perturbations are a matter for the theory of hydrodynamics stability and we refer the reader to [36,210] for a more comprehensive discussion and accurate bibliography on the subject. In the following pages we would like to concentrate only on the results that are closely related to the approach for the Navier–Stokes equations introduced in [34].

Let us start with the L^3-valued mild solutions. First of all, we should note that the application that associates with the initial value $v_0 \in L^3$ the corresponding solution $v(t, x) \in \mathcal{C}([0, T); L^3)$ constructed, as in Kato's theory, by the fixed point theory, is analytical in a neighborhood of zero, as a functional acting on L^3 with values in $\mathcal{C}([0, T); L^3)$, as recalled for instance in [3]. Accordingly, the stability of mild solutions follows immediately because, by virtue of the uniqueness theorem (Section 5), any mild solution arises from the fixed point algorithm. As we will see in Section 7.2, this does not hold the case for the subcritical case $2 \leqslant p < 3$ [41,165].

Generalizing previous stability results in L^p (see [195,228]), Kawanago proceeds in the opposite direction [120,121]. First, he obtains a stability estimate, then makes use of it to establish a uniqueness theorem for mild solution. His result concerns global solutions $v \in \mathcal{C}([0, \infty); L^3)$ and reads as follows. For any $v_0 \in L^3$, there exist two constants $\delta(v_0) > 0$ and $C > 0$ such that, if $\|v_0 - \tilde{v}_0\|_3 < \delta$, then $\tilde{v} \in \mathcal{C}([0, \infty); L^3)$ and

$$\left\| v(t) - \tilde{v}(t) \right\|_3 \leqslant \left\| v(0) - \tilde{v}(0) \right\|_3 \exp\left\{ C \int_0^t \left\| v(s) \right\|_5^5 \mathrm{d}s \right\} \tag{266}$$

for any $t > 0$. Finally, Barraza obtains some stability and uniqueness results for solutions in $L^{(3,\infty)}$ [5]. But, as we have already remarked, the theorem by Meyer in the same Lorentz space [166] allows a considerable simplification of these results.

As pointed out by Yudovich in [239], the choice of the norm for proving the stability of an infinite-dimensional system (e.g., a viscous fluid) is crucial because the Banach norms are not necessarily equivalent therein. To be more explicit, let us recall the simple example of the linear Cauchy problem [85,239]

$$\begin{aligned} &\frac{\partial v}{\partial t} = x \frac{\partial v}{\partial x}, \\ &v(0, x) = \varphi(x), \end{aligned} \tag{267}$$

whose unique (for an arbitrary smooth initial function φ) explicit solution $v(t, x) = \varphi(x \exp(t))$ is exponentially asymptotically stable in $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$, stable but not asymptotically stable in $L^\infty(\mathbb{R})$ or $W^{1,1}(\mathbb{R})$ and exponentially unstable in any $W^{k,p}(\mathbb{R})$ for $k > 1$, $p \geqslant 1$ or $k = 1$, $p > 1$.

7.1. *Lyapunov functions*

A sufficient condition for a solution to be stable for a given norm is that $\|v(t,x) - \tilde{v}(t,x)\|$, the norm of the difference between the solution v and a perturbation $\tilde{v}$, is a decreasing-in-time function. This leads to the following definition.

DEFINITION 12. Let v be a solution of the Navier–Stokes equations, then any decreasing-in-time function $\mathcal{L}(v)(t)$ is called a Lyapunov function associated to v.

The most well-known example is certainly provided by energy

$$E(v)(t) = \frac{1}{2}\|v(t)\|_2^2, \tag{268}$$

because, a calculation similar to the one performed in (251), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\nu\|\nabla v(t)\|_2^2 < 0. \tag{269}$$

This result can easily be generalized in the homogeneous Sobolev spaces $\dot{H}^s$, for $0 \leqslant s \leqslant 1$. For example, in the case $s = \frac{1}{2}$, by means of Hölder and Sobolev inequalities in $\mathbb{R}^3$, we get ([117], p. 258)

$$\big\|\mathbb{P}(v\cdot\nabla)v\big\|_2 \leqslant C\|v\|_6\|\nabla v\|_3 \leqslant C\|v\|_{\dot{H}^1}\|v\|_{\dot{H}^{3/2}}. \tag{270}$$

From this estimate we easily deduce the decreasing property for the function $v = v(t)$ that reads as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\dot{H}^{1/2}}^2 \leqslant -2\|v(t)\|_{\dot{H}^{3/2}}^2\big(\nu - C\|v(t)\|_{\dot{H}^{1/2}}\big) \tag{271}$$

and thus, if the Reynolds number $\|v_0\|_{\dot{H}^{1/2}}/\nu$ is sufficiently small, we get a Lyapunov function associated with the norm $\dot{H}^{1/2}$. As already stated in Section 4.2, a similar argument allows us to obtain for the $\dot{H}^1$ norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\dot{H}^1}^2 \leqslant -2\|v(t)\|_{\dot{H}^2}^2\big(\nu - C\|v(t)\|_{\dot{H}^{1/2}}\big). \tag{272}$$

This estimate shows that the smallness of the number $\|v_0\|_{\dot{H}^{1/2}}/\nu$ also implies the decrease in time of $\|v\|_{\dot{H}^1}$. Now, the Sobolev spaces $\dot{H}^s$, $s > 1/2$ are super-critical. In other words, as far as the scaling is concerned, they have the same invariance as the Lebesgue spaces L^p if $p > 3$. This means that one can prove the existence of a local mild solution for arbitrary initial data (Theorem 1). In the case of $\dot{H}^1$, this solution turns out to be global, provided the quantity $\|v_0\|_{\dot{H}^{1/2}}/\nu$ is sufficiently small, thanks to the uniform estimate

$$\big\|v(t)\big\|_{\dot{H}^1} \leqslant \|v_0\|_{\dot{H}^1} \quad \forall t > 0, \tag{273}$$

that is derived directly from (272).

In other words, this property establishes a direct link between the Lyapunov functions, the existence of global regular solutions in an energy space and the oscillatory behavior of the corresponding initial data.

In a paper that seems to have been completely ignored [115], Kato, after treating the classical cases $\dot{H}^s$, $0 \leqslant s \leqslant 1$, derives new Lyapunov functions for the Navier–Stokes equations not necessarily arising from an energy norm. More precisely: there exists $\delta > 0$ such that if the Reynolds number $R_3(v_0) = \|v_0\|_3/\nu < \delta$, then the quantity $R_3(v)(t) = \|v(t)\|_3/\nu$ is a Lyapunov function associated with v. The importance of this result comes from its connection with the stability theory. In fact, as explained by Joseph [112]:

> It is sometimes possible to find positive definite functionals of the disturbance of a basic flow, other than energy, which decrease on the solutions when the viscosity is larger than a critical value. Such functionals, which may be called generalized energy functionals of the Lyapunov type, are of interest because they can lead to a larger interval of viscosities on which global stability of the basic flow can be guaranteed.

As we proved in [49,50], Kato's result also applies to other functional norms, in particular the Besov ones. See also [2,95,96,145] for related results in this direction. Not only do these properties show the stability for Navier–Stokes in very general functional frames (and imply in particular that set of global regular solutions is open), but as we have noted above, they could shed some light on the research of global Navier–Stokes solutions in supercritical spaces.

7.2. *Dependence on the initial data*

Before leaving this section, we would like to recall a result obtained by Meyer and announced at the Conference in honor of Jacques-Louis Lions held in Paris in 1998 [165]. The full proof will appear in detail in [167]. The theorem in question expresses the dependence on the initial data of the solutions to Navier–Stokes in the subcritical case and could shed some light on the conjecture formulated by Kato in [116], that we recalled in Section 6.3. The result is the following:

THEOREM 19. *There is no application of class $\mathcal{C}^2$ that associates a* (*mild or weak*) *solution $v(t, x) \in \mathcal{C}([0, T); L^p)$, $2 \leqslant p < 3$ to the corresponding initial condition $v_0 \in L^p$.*

Note that $p = 2$ corresponds to the most interesting case of weak solutions by Leray. In particular:

1. There is no application of class $\mathcal{C}^2$ that associates Leray's weak solution $v(t, x) \in L^\infty((0, T); L^2)$, to the initial condition $v_0 \in L^2$.
2. If a mild solution exists in the subcritical case ($2 \leqslant p < 3$), it does not arise from a fixed point algorithm. On the other hand, as we have seen in Section 7, the application that associates Kato's mild solution $v(t, x) \in \mathcal{C}([0, T); L^3)$ to the initial data $v_0 \in L^3$ is analytical in a neighborhood of zero as a functional acting on L^3 and taking values in $\mathcal{C}([0, T); L^3)$. In the subcritical case, the regularity of the flow-map changes drastically.

The proof of Theorem 19 is based on a contradiction argument. Briefly stated, it is assumed that for the initial data λv_0, the solution $v_\lambda(t, x)$, whose existence is supposed in Theorem 19, could be written in the form $\lambda v^{(1)}(t, x) + \lambda^2 v^{(2)}(t, x) + o(\lambda^2)$, where little o corresponds to the norm $L^\infty([0, T); L^p)$ and $\lambda \to 0$. Then, the idea is to evaluate (by calculations analogous to that performed in Section 3.4.2) the norm of the bilinear operator that defines $v^{(2)}(t, x)$ in terms of v_0 in order to prove that $v^{(2)}(t, x)$ cannot belong to $\mathcal{C}([0, T); L^p)$. As usual, the main point will be to evaluate not the "exact" value of the symbol of the operator, but its "homogeneity scaling".

This kind of ill-posedness results for solutions arising from the Banach fixed point theorem in the case of the Navier–Stokes equations can be easily generalized to the nonlinear heat equation, the viscous Hamilton–Jacobi equation and the convection–diffusion equation, as it is illustrated in the paper [41].

Conclusion

Should we conclude from the three examples given in this paper (oscillations, uniqueness and self-similarity) that real variable methods are *always* better suited for the study of Navier–Stokes, and that wavelets, paraproducts, Littlewood–Paley decomposition, Besov spaces and harmonic analysis tools in general have nothing to do with these equations?

In order to analyze this question, we list here a series of bad and good news, that will be summarized by a prophetic wish.

For the Navier–Stokes equations, there are other examples in which Fourier methods do not gain against real variable methods. For example, by using Fourier transform in [109], Heywood was hoping to get a better global estimate for $\|\nabla v(t)\|_2$, in order to improve the key inequality analyzed in Section 5.1, Equation (220). However, as he remarks in [110]:

> We give Fourier transform estimates for solutions of the Navier–Stokes equations, without using Sobolev's inequalities, getting again global existence in two dimensions but only local existence in three dimensions. [...] Unfortunately, because of a dimensional dependence in the evaluation of a singular integral, the final result is only a local existence theorem in the three-dimensional. [...] This adds another failure to an already long list of failures to prove global existence in the three-dimensional case, which may reinforce the feeling that singularities really exist.

In practical applications, one never looks for a solution in $\mathbb{R}^3$, yet solid bodies (e.g., the surface of a container), limit the region of space where the flow takes place. However, in the physically more interesting case when boundaries are present, it is very difficult to generalize the methods based on Fourier transform techniques (see [51,68,69,153,154,175,240]), unless some periodicity conditions are considered, like, e.g., the torus $\mathbb{T}^3$ (see [222]).

The situation seems more favorable to Fourier methods in the case of decay as $t \to \infty$ of solutions of the Navier–Stokes equations (see [23–27,95,96,237]). So far, no better techniques than the Fourier splitting introduced by Schonbek and the Hardy spaces considered by Miyakawa [171–173] are known to study the decay at infinity of solutions to the Navier–Stokes equations.

Finally, in the case of the Euler equations, there is a rich literature that makes use of paradifferential tools (see [55,61,229–231]). However, in the case of vortex patches, whose regularity was proved in 1993 by Chemin using Bony's paraproduct rule (see [57,61]),

a much simpler proof that does not make use of the paradifferential machinery was discovered by Bertozzi and Constantin [11,12] and by Serfati [207].

The discussion seems endless, the examples innumerable and it is difficult to conclude. As announced, we will to do it with a messianic hope of Federbush [79]:

> One should be able to do more than we have accomplished so far using wavelets: make a dent in the question of the existence of global strong solutions, find a theoretical formalism for turbulence [...]. Someone (perhaps smarter than me, perhaps working harder than me, perhaps luckier than me, perhaps younger than me) should get much further on turbulence and the Navier–Stokes equations with the ideas in wavelet analysis.

Acknowledgments

References

[1] H. Amann, *On the strong solvability of the Navier–Stokes equations*, J. Math. Fluid Mech. **2** (2000), 16–98.

[2] P. Auscher, S. Dubois and P. Tchamitchian, *On the stability of global solutions to Navier–Stokes equations in the space*, J. Math. Pures Appl. **83** (6) (2004), 673–697.

[3] P. Auscher and P. Tchamitchian, *Espaces critiques pour le système des équations de Navier–Stokes incompressibles*, preprint, Université de Picardie Jules Verne (1999); available online at http://www.mathinfo.u-picardie.fr/auscher/.

[4] O.A. Barraza, *Self-similar solutions in weak L^p spaces of the Navier–Stokes equations*, Rev. Mat. Iberoamericana **12** (1996), 411–439.

[5] O.A. Barraza, *Regularity and stability for the solutions of the Navier–Stokes equations in Lorentz spaces*, Nonlinear Anal. **35** (6) (1999), 747–764.

[6] G. Battle and P. Federbush, *Divergence-free vector wavelets*, Michigan Math. J. **40** (1995), 181–195.

[7] G. Battle, P. Federbush and P. Uhlig, *Wavelets for quantum gravity and divergence-free wavelets*, Appl. Comput. Harmon. Anal. **1** (1994), 295–297.

[8] R.N. Bhattacharya, L. Chen, S. Dobson, R.B. Guenther, C. Orum, M. Ossiander, E. Thomann and E.C. Waymire, *Majorizing kernels* & *stochastic cascades with applications to incompressible Navier–Stokes equations*, Trans. Amer. Math. Soc. **355** (12) (2003), 5003–5040 (electronic).

[9] M. Ben-Artzi, *Global solutions of two-dimensional Navier–Stokes and Euler equation*, Arch. Ration. Mech. Anal. **128** (1994), 329–358.

[10] J. Bergh and J. Löfstrom, *Interpolation Spaces*, *an Introduction*, Springer-Verlag, Berlin (1976).

[11] A.L. Bertozzi and P. Constantin, *Global regularity of vortex patches*, Comm. Math. Phys. **152** (1993), 19–28.

[12] A.L. Bertozzi and A.J. Majda, *Vorticity and Incompressible Flow*, Cambridge Texts Appl. Math., Cambridge Univ. Press (2002).

[13] O.V. Besov, *Investigation of a Family of Functional Spaces Connected with Embedding and Extension Theorems*, Trudy Mat. Inst. Steklov **60** (1961) (in Russian).

[14] H.A. Biagioni, L. Cadeddu and T. Gramchev, *Semilinear parabolic equations with singular initial data in anisotropic weighted spaces*, Differential Integral Equations **12** (1999), 613–636.

[15] H.A. Biagioni and T. Gramchev, *Evolution PDE with elliptic dissipative terms: Critical index for singular initial data, self-similar solutions and analytic regularity*, C. R. Acad. Sci. Paris Sér. I Math. **327** (1) (1998), 41–46.

[16] P. Biler, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation*, Stud. Math. **114** (1995), 181–205.

[17] P. Biler, *Local and global solutions of a nonlinear nonlocal parabolic problem* (Warsaw, 1994), Gakuto Internat. Ser. Math. Sci. Appl., Vol. 7, Gakkōtosho, Tokyo (1996), 49–66.

[18] P. Biler, M. Cannone, I. Guerra and G. Karch, *Global regular and singular solutions for a model of gravitating particles*, Math. Ann. (2004), to appear.

[19] P. Biler, M. Cannone and G. Karch, *Asymptotic Stability of the Navier–Stokes Flow Past an Obstacle*, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences, Warszawa (2004), to appear.

[20] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. Ecole Norm. Sup. **14** (4) (1981), 209–246.

[21] G. Bourdaud, *Réalisation des espaces de Besov homogènes*, Ark. Mat. **26** (1) (1988), 41–54.

[22] G. Bourdaud, *Ce qu'il faut savoir sur les espaces de Besov*, Prépublication de l'Université de Paris 7 (Janvier 1993).

[23] L. Brandolese, *Loss of the localization of asymmetric solutions of the Navier–Stokes equations in $\mathbb{R}^n$*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2) (2001), 125–130.

[24] L. Brandolese, *Localisation, Oscillations et Comportement Asymptotique pour les Equations de Navier–Stokes*, Doctoral Thesis, Ecole Normale Supérieure de Cachan, France (2001).

[25] L. Brandolese, *Asymptotic behavior of the energy and pointwise estimates for solutions to the Navier–Stokes equations*, Rev. Mat. Iberoamericana **20** (1) (2004), 223–256.

[26] L. Brandolese, *Atomic decomposition for the vorticity of a viscous fluid*, Math. Nachr., to appear.

[27] L. Brandolese and Y. Meyer, *On the instantaneous spreading for the Navier–Stokes system in the whole space*, ESAIM Control Optim. Calc. Var. **8** (2002), 273–285 (electronic).

[28] H. Brezis, *Remarks on the preceding paper by M. Ben-Artzi*, Arch. Ration. Mech. Anal. **128** (4) (1994), 359–360.

[29] H. Brezis and T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. **68** (1996), 277–304.

[30] F.E. Browder, *Non-linear equations of evolution*, Ann. Math. **80** (1964), 485–523.

[31] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771–837.

[32] C.P. Calderón, *Existence of weak solutions for the Navier–Stokes equations with initial data in L^p*, Trans. Amer. Math. Soc. **318** (1) (1990), 179–200; Addendum, ibid., 201–207.

[33] C.P. Calderón, *Initial values of solutions of the Navier–Stokes equations*, Proc. Amer. Math. Soc. **117** (3) (1993), 761–766.

[34] M. Cannone, *Ondelettes, paraproduits et Navier–Stokes*, Diderot, Paris (1995).

[35] M. Cannone, *A generalization of a theorem by Kato on Navier–Stokes equations*, Rev. Mat. Iberoamericana **13** (3) (1997), 515–541.

[36] M. Cannone, *Nombres de Reynolds, stabilité et Navier–Stokes*, Evolution Equations: Existence and Singularities, Vol. 52, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences, Warszawa (2000), 29–59.

[37] M. Cannone, *Viscous flows in Besov spaces*, Proc. Sixth Paseki School: Mathematical Theory in Fluid Mechanics, J. Málek, J. Nečas and M. Rokyta, eds, Springer-Verlag, Berlin (2000), 1–34.

[38] M. Cannone and S. Friedlander, *Navier: Blow-up and collapse*, Amer. Math. Soc. Notices **50** (1) (2003), 7–13.

[39] M. Cannone and G. Karch, *Incompressible Navier–Stokes equations in abstract Banach spaces*, Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics, H. Fujita, S.T. Kuroda and H. Okamoto, eds, Yurinsha, Tokyo (2002), 27–41; appeared also in Sūrikaisekikenkyūsho Kōkyūroku **1234** (2001).

[40] M. Cannone and G. Karch, *Smooth or singular solutions to the Navier–Stokes system?*, J. Differential Equations **197** (2) (2003), 247–274.

[41] M. Cannone and G. Karch, *On the validity of Picard algorithm for nonlinear parabolic equations*, Preprint (2003).

[42] M. Cannone and G. Karch, *About the generalized Navier–Stokes equations*, J. Math. Fluid Mech. (2004), to appear.

[43] M. Cannone and Y. Meyer, *Littlewood–Paley decomposition and the Navier–Stokes equations*, Meth. Appl. Anal. **2** (1995), 307–319.

[44] M. Cannone, Y. Meyer and F. Planchon, *Solutions auto-similaires des équations de Navier–Stokes in* $\mathbb{R}^3$, Exposé n. VIII, Séminaire X-EDP, Ecole Polytechnique (1994).

[45] M. Cannone and F. Planchon, *Self-similar solutions of the Navier–Stokes equations in* $\mathbb{R}^3$, Comm. Partial Differential Equations **21** (1996), 179–193.

[46] M. Cannone and F. Planchon, *Maximal function inequalities and Navier–Stokes equations*, *Investigations on the structure of solutions to partial differential equations* (in Japanese) (Kyōto, 1997), Sūrikaisekikenkyūsho Kōkyūroku **1036** (1998), 139–159.

[47] M. Cannone and F. Planchon, *On the non stationary Navier–Stokes equations with an external force*, Adv. Differential Equations **4** (5) (1999), 697–730.

[48] M. Cannone and F. Planchon, *On the regularity of the bilinear term for solutions of the incompressible Navier–Stokes equations in* $\mathbb{R}^3$, Rev. Mat. Iberoamericana **1** (2000), 1–16.

[49] M. Cannone and F. Planchon, *Fonctions de Lyapunov pour les équations de Navier–Stokes*, Exposé n. VIII, Séminaire X-EDP, Ecole Polytechnique (2000), 1–7.

[50] M. Cannone and F. Planchon, *More Lyapunov functions for the Navier–Stokes equations*, Navier–Stokes equations: Theory and Numerical Methods, R. Salvi, ed., Lectures Notes in Pure and Appl. Math., Vol. 223, Dekker, New York (2001), 19–26.

[51] M. Cannone, F. Planchon and M.E. Schonbek, *Navier–Stokes equations in the half space*, Comm. Partial Differential Equations **25** (5/6) (2000), 903–924.

[52] T. Cazenave and F.B. Weissler, *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations*, Math. Z. **228** (1) (1998), 83–120.

[53] T. Cazenave and F.B. Weissler, *More self-similar solutions of the nonlinear Schrödinger equation*, NoDEA Nonlinear Differential Equations Appl. **5** (3) (1998), 355–365.

[54] T. Cazenave and F.B. Weissler, *Scattering theory and self-similar solutions for the nonlinear Schrödinger equation*, SIAM J. Math. Anal. **31** (2000), 625–650 (electronic).

[55] D. Chae, *On the well-posedness of the Euler equations in the Triebel–Lizorkin spaces*, Comm. Pure Appl. Math. **55** (5) (2002), 654–678.

[56] J.-Y. Chemin, *Remarques sur l'existence globale pour le système de Navier–Stokes incompressible*, SIAM J. Math. Anal. **23** (1992), 20–28.

[57] J.-Y. Chemin, *Persistence de structures géometriques dans les fluides incompressibles bidimensionnels*, Ann. Sci. École Norm. Sup. **26** (1993), 1–16.

[58] J.-Y. Chemin, *About Navier–Stokes system*, Publ. Lab. Anal. Num. (Univ. Paris 6) **R 96023** (1996), 1–43.

[59] J.-Y. Chemin, *Sur l'unicité dans le système de Navier–Stokes tridimensionnel*, Exposé n. XXIV, Séminaire X-EDP, Ecole Polytechnique (1996–1997).

[60] J.-Y. Chemin, *Théorèmes d'unicité pour le système de Navier–Stokes tridimensionnel*, J. Anal. Math. **77** (1997), 27–50.

[61] J.-Y. Chemin, *Fluides Parfaits Incompressibles*, Astérisque **230** (1995); *Perfect Incompressible Fluids*, transl. by I. Gallagher and D. Iftimie, Oxford Lecture Series in Mathematics and Its Applications, Vol. 14, Clarendon Press–Oxford University Press, New York (1998).

[62] Z.-M. Chen, L_n *solutions to the stationary and nonstationary Navier–Stokes equations in* $\mathbb{R}^n$, Pacific J. Math. **158** (1993), 293–303.

[63] Z.-M. Chen and Z. Xin, *Homogeneity criterion on the Navier–Stokes equations in the whole spaces*, J. Math. Fluid Mech. **3** (2) (2001), 152–182.

[64] Z.-M. Chen, Y. Zhao and Y. Yang, *Hölder–Zygmund techniques to the Navier–Stokes equations in the whole spaces*, J. Partial Differential Equations **13** (2000), 89–96.

[65] H.J. Choe and H. Kim, *Isolated singularity for the stationary Navier–Stokes system*, J. Math. Fluid Mech. **2** (2) (2000), 151–184.

[66] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[67] P. Constantin, *A few results and open problems regarding incompressible fluids*, Notices Amer. Math. Soc. **42** (6) (1995), 658–663.

[68] N. Depauw, *Solutions peu régulières des equations d'Euler et Navier–Stokes incompressibles sur un domaine à bord*, Doctoral Thesis, Université de Paris Nord, France (1998).

[69] N. Depauw, *Solutions des équations de Navier–Stokes incompressibles dans un domaine extérieur*, Rev. Mat. Iberoamericana **17** (1) (2000), 21–68.

[70] R.J. DiPerna, P.-L. Lions and Y. Meyer, *L^p-regularity of velocity averages*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), 271–287.

[71] S. Dubois, *What is a solution to the Navier–Stokes equations?*, C. R. Acad. Sci. Paris **335** (2002), 27–32.

[72] S. Dubois, *Equations de Navier–Stokes dans l'Espace: Espaces Critiques et Solutions d'Énergie Finie*, Doctoral Thesis, Université de Picardie Jules Verne (2002).

[73] S. Dubois, *Mild solutions to the Navier–Stokes equations and energy equalities*, J. Differential Equations **189** (1) (2003), 99–147.

[74] J. Earman, *A Primer on Determinism*, Reidel, Dordrecht (1986).

[75] L. Escauriaza, G. Seregin and V. Sverák, *On $L^{3,\infty}$-solutions to the Navier–Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk **58** (2) (2003), 3–44.

[76] E.B. Fabes, B.F. Jones and N.M. Rivière, *The initial value problem for the Navier–Stokes equations with data in L^p*, Arch. Ration. Mech. Anal. **45** (1972), 222–240.

[77] M. Farge, *Wavelets transform and their applications to turbulence*, Ann. Rev. Fluid Mech. **24** (1992), 395–457.

[78] P. Federbush, *Navier and Stokes meet the wavelet*, Comm. Math. Phys. **155** (1993), 219–248.

[79] P. Federbush, *Navier and Stokes meet the wavelet, II*, Mathematical Quantum Theory. I. Field Theory and Many-Body Theory, Vancouver, BC, 1993, CRM Proc. Lecture Notes **7** (1994), 163–169.

[80] C.L. Fefferman, *Existence and uniqueness of the Navier–Stokes equation* (2002); available online at http://www.claymath.org/millennium/Navier–Stokes_Equations.

[81] C. Foias and R. Temam, *Self-similar universal homogeneous statistical solutions of the Navier–Stokes equations*, Comm. Math. Phys. **90** (1983), 187–206.

[82] M. Frazier, B. Jawerth and G. Weiss, *Littlewood–Paley Theory and the Study of Function Spaces*, CBMS Reg. Conf. Ser. Math., Vol. 79, Amer. Math. Soc., Providence, RI (1991).

[83] J. Frehse and J. Málek, *Problems due to the no-slip boundary in incompressible fluid dynamics*, Geometric Analysis and Nonlinear Partial Differential Equations, S. Hildebrandt and H. Karcher, eds, Springer-Verlag (2003), 559–571.

[84] S. Friedlander and N. Pavlović, *Remarks concerning a modified Navier–Stokes equation*, Discrete Contin. Dyn. Syst. **10** (1/2) (2004), 269–288.

[85] S. Friedlander and V. Yudovich, *Instabilities in fluid motion*, Notices Amer. Math. Soc. **46** (11) (1999), 1358–1367.

[86] J. Fourier, *Théorie analytique de la chaleur*, 1822, published in Œuvres de Fourier **1** Gauthiers-Villar, Paris (1888).

[87] H. Fujita and T. Kato, *On the Navier–Stokes initial value problem, I*, Arch. Ration. Mech. Anal. **16** (1964), 269–315.

[88] H. Fujita, S.T. Kuroda and H. Okamoto, eds, *Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics*, Yurinsha, Tokyo (2002); appeared also in Sūrikaisekikenkyūsho Kōkyūroku **1234** (2001).

[89] G. Furioli, *Applications de l'analyse harmonique réelle à l'etude des equations de Navier–Stokes et de Schrödinger non linéaire*, Doctoral Thesis, Université de Paris Sud, France (1999).

[90] G. Furioli, *On the existence of selfsimilar solutions of the nonlinear Schrödinger equation with power nonlinearity between* 1 *and* 2, Differential Integral Equations **14** (10) (2001), 1259–1266.

[91] G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo, *Sur l'unicité dans $L^3(\mathbb{R}^3)$ des solutions mild des équations de Navier–Stokes*, C. R. Acad. Sci. Paris Sér. I Math. **325** (12) (1997), 1253–1256.

[92] G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo, *Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces fonctionnels limites pour Navier–Stokes*, Rev. Mat. Iberoamericana **16** (3) (2000), 605–667.

[93] G. Furioli, P.-G. Lemarié-Rieusset, E. Zahrouni and A. Zhioua, *Un théorème de persistence de la régularité en norme d'espaces de Besov pour les solutions de Koch and Tataru des équations de Navier–Stokes dans $\mathbb{R}^3$*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 339–342.

[94] G. Furioli and E. Terraneo, *Molecules of the Hardy space and the Navier–Stokes equations*, Funkcial. Ekvac. **45** (1) (2002), 141–160.

[95]   I. Gallagher, D. Iftimie and F. Planchon, *Non-explosion en temps grand et stabilité de solutions globales des équations de Navier–Stokes*, C. R. Acad. Sci. Paris Sér. I Math. **334** (4) (2002), 289–292.

[96]   I. Gallagher, D. Iftimie and F. Planchon, *Asymptotics and stability for global solutions to the Navier–Stokes equations*, Ann. Inst. Fourier **53** (5) (2003), 1387–1424.

[97]   I. Gallagher and F. Planchon, *On global infinite energy solutions to the Navier–Stokes equations in two dimensions*, Arch. Ration. Mech. Anal. **161** (2002), 307–337.

[98]   P. Gérard, Y. Meyer and F. Oru, *Inégalités de Sobolev précisées*, *Exposé n. VIII*, Séminaire X-EDP, Ecole Polytechnique (1996–1997).

[99]   Y. Giga, *Regularity criteria for weak solutions of the Navier–Stokes system*, Proc. Symp. Pure Math. **45** (1) (1986), 449–453.

[100] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system*, J. Differential Equations **62** (1986), 186–212.

[101] Y. Giga, *Review of the paper by H. Brezis*, "Remarks on the preceding paper by M. Ben-Artzi", Math. Rev. **96h:35149** (1996); see, for the revised form, at the MathSciNet http address.

[102] Y. Giga and M.-H. Giga, *Nonlinear Partial Differential Equations – Asymptotic Behaviour of Solutions and Self-Similar Solutions* (in Japanese), Kyōritsu Shuppan, Tokyo (1999), English transl. to appear.

[103] Y. Giga, K. Inui, J. Kato and S. Matsui, *Remarks on the uniqueness of bounded solutions of the Navier–Stokes equations*, Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000), Nonlinear Anal. **47** (6) (2001), 4151–4156.

[104] Y. Giga, K. Inui and S. Matsui, *On the Cauchy Problem for the Navier–Stokes Equations with Nondecaying Initial Data*, Adv. Fluid Dynamics, Quad. Mat., Vol. 4, P. Maremonti, ed., Aracne, Roma (1999), 28–68.

[105] Y. Giga, S. Matsui and O. Sawada, *Global existence of two-dimensional Navier–Stokes flow with nonde-caying initial velocity*, J. Math. Fluid Mech. **3** (2001), 302–315.

[106] Y. Giga and T. Miyakawa, *Solutions in L^r of the Navier–Stokes initial value problem*, Arch. Ration. Mech. Anal. **89** (1985), 267–281.

[107] Y. Giga and T. Miyakawa, *Navier–Stokes flows in $\mathbb{R}^3$ with measures as initial vorticity and the Morrey spaces*, Comm. Partial Differential Equations **14** (1989), 577–618.

[108] A. Haraux and F.B. Weissler, *Non-uniqueness for a semi-linear initial value problem*, Indiana Univ. Math. J. **31** (1985), 167–189.

[109] J.G. Heywood, *Fourier transform estimates for the Navier–Stokes equations*, Proc. Internat. Conf. Advances in Geometric Analysis and Continuum Mechanics, Standford University, International Press, Cambridge (1993), 109–115.

[110] J.G. Heywood, *Remarks on the possible global regularity of solutions of the three-dimensional Navier–Stokes equations*, Progress in Theoretical and Computational Fluid Mechanics, Paseky 1993, Pitman Res. Notes Math. Ser., Vol. 308, Longman, Harlow (1994), 1–32.

[111] R. Hunt, *On $L(p,q)$ spaces*, L'Enseignement Mathématique **12** (4) (1966), 249–276.

[112] D.D. Joseph, *Stability of Fluid Motions* (2 Vols), Springer Tracts in Natural Philosophy, Springer-Verlag, Berlin (1976).

[113] G. Karch, *Scaling in nonlinear parabolic equations*, J. Math. Anal. Appl. **234** (1999), 534–558.

[114] T. Kato, *Strong L^p solutions of the Navier–Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Math. Z. **187** (1984), 471–480.

[115] T. Kato, *Liapunov functions and monotonicity in the Navier–Stokes equation*, Lecture Notes in Math., Vol. 1450, Springer-Verlag, Berlin (1990), 53–63.

[116] T. Kato, *Well-posedness nitsuite* (in Japanese), Sūgaku **48** (3) (1996), 298–300.

[117] T. Kato and H. Fujita, *On the non-stationary Navier–Stokes system*, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[118] T. Kato and G. Ponce, *The Navier–Stokes equations with weak initial data*, Int. Math. Res. Notes **10** (1994), 435–444.

[119] N.H. Katz and N. Pavlović, *A cheap Caffarelli–Kohn–Nirenberg inequality for Navier–Stokes equations with hyper-dissipation*, Geom. Funct. Anal. **12** (2) (2002), 355–379.

[120] T. Kawanago, *Stability of global strong solutions of the Navier–Stokes equations*, *Nonlinear evolution equations and their applications* (in Japanese), Sūrikaisekikenkyūsho Kōkyūroku **913** (1995), 141–147.

[121] T. Kawanago, *Stability estimate of strong solutions for the Navier–Stokes system and its applications*, Electron. J. Differential Equations (electronic) **15** (1998), 1–23; available online at http://ejde.math.swt.edu/Volumes/1998/15-Kawanago/abstr.html.

[122] S. Klainerman, *PDE as a unified subject*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal., Special Vol., Part I (2000), 279–315.

[123] H. Koch and D. Tataru, *Well-posedness for the Navier–Stokes equations*. Adv. Math. **157** (1) (2001), 22–35.

[124] H. Kozono, *Well-posedness of the Navier–Stokes equations*, EVEQ 2000 lectures notes (manuscript), Parts I and II, July 3–7 Prague (2000).

[125] H. Kozono and H. Sohr, *Remark on uniqueness of weak solutions to the Navier–Stokes equations*, Analysis **16** (3) (1996), 255–271.

[126] H. Kozono and Y. Taniuchi, *Bilinear estimates in BMO and the Navier–Stokes equations*, Math. Z. **235** (1) (2000), 173–194.

[127] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO*, *with application to the Euler equation*, Comm. Math. Phys. **214** (1) (2000), 191–200.

[128] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier–Stokes equation with distributions as initial data*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), 1127–1132.

[129] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), 959–1014.

[130] H. Kozono and M. Yamazaki, *The stability of small stationary solutions in Morrey spaces of the Navier–Stokes equation*, Indiana Univ. Math. J. **44** (1995), 1307–1336.

[131] H. Kozono and M. Yamazaki, *The exterior problem for the non-stationary Navier–Stokes equation with data in the space $L^{n,\infty}$*, C. R. Acad. Sci. Paris Sér. I Math. **320** (6) (1995), 685–690.

[132] H. Kozono and M. Yamazaki, *The Navier–Stokes equation with distributions as initial data and application to self-similar solutions*, New Trends in Microlocal Analysis (Tokyo, 1995) (1997), 125–141.

[133] H. Kozono and M. Yamazaki, *Exterior problem for the stationary Navier–Stokes equations in the Lorentz space*, Math. Ann. **310** (2) (1998), 279–305.

[134] H. Kozono and M. Yamazaki, *On a larger class of stable solutions to the Navier–Stokes equations in exterior domains*, Math. Z. **228** (1998), 751–785.

[135] L.D. Landau, *A new exact solution of Navier–Stokes equations*, C. R. (Doklady) Acad. Sci. USSR (N.S.) **43** (1944), 286–288.

[136] L.D. Landau and E.M. Lifchitz, *Fluid Dynamics* (in Russian), Nauka, Moscow, 1986; English transl. by J.B. Sykes and W.H. Reid: Course of Theoretical Physics, Vol. 6, Pergamon Press, London–Paris–Frankfurt; Addison-Wesley, Reading, MA (1959).

[137] Y. Le Jan and A.-S. Sznitman, *Cascades aléatoires et équations de Navier–Stokes*, C. R. Acad. Sci. Paris **324** (7) (1997), 823–826.

[138] Y. Le Jan and A.-S. Sznitman, *Stochastic cascades and 3-dimensional Navier–Stokes equations*, Probab. Theory Related Fields **109** (3) (1997), 343–366.

[139] P.-G. Lemarié, *Analyses multi-résolutions non othogonales*, *commutation entre projecteurs et dérivations et ondelettes vecteurs à divergence nulle*, Rev. Mat. Iberoamericana **8** (1992), 221–237.

[140] P.-G. Lemarié, *Un théorème d'inxistence pour les ondelettes vecteurs à divergence nulle*, C. R. Acad. Sci. Paris **319** (8) (1994), 811–813.

[141] P.-G. Lemarié-Rieusset, *Some remarks on the Navier–Stokes equations in $\mathbb{R}^3$*, J. Math. Phys. **39** (8) (1998), 4108–4118.

[142] P.-G. Lemarié-Rieusset, *Solutions faibles d'énergie infinie pour les équations de Navier–Stokes dans $\mathbb{R}^3$*, C. R. Acad. Sci. Paris Sér. I **328** (8) (1999), 1133–1138.

[143] P.-G. Lemarié-Rieusset, *Cinq petits théorèmes d'unicité L^3 des solutions des équations de Navier–Stokes dans $\mathbb{R}^3$*, Colloque equations de Navier–Stokes et analyse microlocale, CIRM, Luminy, Marseille lectures notes (manuscript) (9–13 Novembre 1999), 1–9.

[144] P.-G. Lemarié-Rieusset, *Une remarque sur l'analyticité des solutions milds des équations de Navier–Stokes dans $\mathbb{R}^3$*, C. R. Acad. Sci. Paris Sér. I **330** (2000), 183–186.

[145] P.-G. Lemarié-Rieusset, *Recent Developments in the Navier–Stokes Problem*, Chapman & Hall/CRS Res. Notes Math., Chapman & Hall/CRC, Boca Raton, FL (2002).

[146] P.-G. Lemarié-Rieusset, *Espaces de Lorentz et Navier–Stokes: Le problème des solutions auto-similaires de Leray*, Preprint, Université d'Evry Val d'Essonne (2002); available online at http://www.maths.univ-evry.fr/pages_perso/lemarie/mypub.html.

[147] P.-G. Lemarié-Rieusset and R. May, *Uniqueness for the Navier–Stokes equations and multipliers between Sobolev spaces*, Preprint of the Université d'Evry Val d'Essonne (2002).

[148] J. Leray, *Etudes de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*, J. Math. Pures Appl. **12** (1933), 1–82.

[149] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.

[150] J. Leray, *Aspects de la mécanique théorique des fluides*, La Vie des Sciences, C. R. Sér. Générale II **11** (4) (1994), 287–290.

[151] F.-H. Lin, *A new proof of the Caffarelli–Kohn–Nirenberg theorem*, Comm. Pure Appl. Math. **51** (1998), 240–257.

[152] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, Clarendon Press, Oxford, Vol. 1 (1996); Vol. 2 (1998).

[153] P.-L. Lions and N. Masmoudi, *Unicité des solutions faibles de Navier–Stokes dans $L^N(\Omega)$*, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), 491–496.

[154] P.-L. Lions and N. Masmoudi, *Uniqueness of mild solutions of the Navier–Stokes system in L^N*, Comm. Partial Differential Equations **26** (11–12) (2001), 2211–2226.

[155] J. Málek, J. Nečas, M. Pokorný and M.E. Schonbek, *On possible singular solutions to the Navier–Stokes equations*, Math. Nachr. **199** (1999), 97–114.

[156] A.L. Mazzucato, *Analysis of the Navier–Stokes and other non-linear evolution equations with initial data in Besov-type spaces*, Ph.D. Thesis, University of North Carolina at Chapel Hill (2000).

[157] A.L. Mazzucato, *Besov–Morrey spaces: Function space theory and applications to non-linear PDE*, Trans. Amer. Math. Soc. **355** (4) (2003), 1297–1364 (electronic).

[158] A.L. Mazzucato, *Decomposition of Besov–Morrey spaces*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), Contemp. Math., Vol. 320, Amer. Math. Soc., Providence, RI (2003), 279–294.

[159] R. May, *Existence, Unicité et Régularité des Solutions Faibles des Équations de Navier–Stokes*, Doctoral Thesis, Université d'Evry Val d'Essonne, France (1999).

[160] R. May, *Rôle de l'espace de Besov $B_\infty^{-1,\infty}$ dans le contrôle de l'explosion éventuelle en temps fini des solutions régulières des équations de Navier–Stokes*, C. R. Acad. Sci. Paris Sér. I Math. **336** (9) (2003), 731–734.

[161] Y. Meyer, *La Minimalité de l'Espace de Besov $\dot{B}_1^{0,1}$ et la Continuité des Opérateurs Définis par des Intégrales Singulières*, Univ. Autonoma de Madrid, Monografias de Matematicas, Vol. 4 (1986).

[162] Y. Meyer, *Wavelets and Operators*, Cambridge Studies in Advanced Mathematics, Vol. 37, Cambridge Univ. Press, Cambridge (1993).

[163] Y. Meyer, *Wavelets and fast numerical algorithms*, Handbook of Numerical Analysis, Vol. 5, P.-G. Ciarlet and J.-L.Lions, eds, Elsevier, Amsterdam (1997), 639–713.

[164] Y. Meyer, *Large time behavior and self-similar solutions of some semi-linear diffusion equations*, Harmonic Analysis and Partial Differential Equations (Chicago, IL, 1996), Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL (1999), 241–261.

[165] Y. Meyer, *New estimates for Navier–Stokes equations* (*two manuscripts*), Colloque en l'Honneur de J.-L. Lions à l'Occasion de Son 70e Anniversaire, Paris 26–27 mai 1998, Auditorium du CNRS, 3 rue Michel-Ange, Paris XVIe (1998).

[166] Y. Meyer, *Wavelets, paraproducts and Navier–Stokes equations*, Current Developments in Mathematics 1996, International Press, Cambridge, MA (1999), 105–212.

[167] Y. Meyer, *The role of oscillations in some nonlinear problems*. European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., Vol. 201, Birkhäuser, Basel (2001), 75–99.

[168] Y. Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, The Fifteenth Dean Jacqueline B. Lewis Memorial Lectures, University Lecture Series, Vol. 22, Amer. Math. Soc., Providence, RI (2001).

[169] Y. Meyer, *Oscillating patterns in some nonlinear evolution equations*, Mathematical Foundation of Turbulent Viscous Flows, Lectures Notes in Math., M. Cannone and T. Miyakawa, eds, Springer-Verlag, New York (2004), to appear.

[170] J.R. Miller, M. O'Leary and M.E. Schonbek, *Nonexistence of singular pseudo-self-similar solutions of the Navier–Stokes system*, Math. Ann. **319** (2001), 809–815.

[171] T. Miyakawa, *Hardy spaces of solenoidal vector fields*, with applications to the Navier–Stokes equations, Kyushu J. Math. **50** (1) (1996), 1–64.

[172] T. Miyakawa, *Application of Hardy space techniques to the time-decay problem for incompressible Navier–Stokes flows in $\mathbb{R}^n$*, Funkcial. Ekvac. **41** (3) (1998), 383–434.

[173] T. Miyakawa, *On space-time decay properties of nonstationary incompressible Navier–Stokes flows in $\mathbb{R}^n$*, Funkcial. Ekvac. **43** (3) (2000), 541–557.

[174] S. Monniaux, *Uniqueness of mild solutions of the Navier–Stokes equation and maximal L^p-regularity*, C. R. Acad. Sci. Paris Sér. I Math. **328** (8) (1999), 663–668.

[175] S. Monniaux, *On Uniqueness for the Navier–Stokes system in 3D-bounded Lipschitz domains*, J. Funct. Anal. **195** (1) (2002), 1–11.

[176] S. Montgomery-Smith, *Finite time blow-up for a Navier–Stokes like equation*, Proc. Amer. Math. Soc. **129** (10) (2001), 3025–3029 (electronic).

[177] S. Montgomery-Smith, *A condition implying regularity of the three dimensional Navier–Stokes equation*, Preprint (2003).

[178] C.L.M.H. Navier, *Mémoire sur les lois du mouvement des fluides*, Mém. Acad. Sci. Inst. France **6** (1822), 389–440.

[179] J. Nečas, M. Růžička and V. Šverák, *Sur une remarque de J. Leray concernant la construction de solutions singulières des équations de Navier–Stokes*, C. R. Acad. Sci. Paris Sér. I Math. **323** (3) (1996), 245–249.

[180] J. Nečas, M. Růžička and V. Šverák, *On Leray self-similar solutions of the Navier–Stokes equations*, Acta Math. **176** (2) (1996), 283–294.

[181] H. Okamoto, *Exact solutions of the Navier–Stokes equations via Leray's scheme*, Proc. Miniconf. Partial Differential Equations and Applications, Seoul Nat. Univ., Seoul (1997); Lecture Notes Ser., Vol. 38, Seoul Nat. Univ. (1997), 1–30.

[182] H. Okamoto, *Exact solutions of the Navier–Stokes equations via Leray's scheme*, Japan J. Indust. Appl. Math. **14** (2) (1997), 169–197.

[183] F. Oru, *Rôle des oscillations dans quelques problèmes d'analyse non linéaire*, Doctoral Thesis, Ecole Normale Supérieure de Cachan, France (1998).

[184] H. Pecher, *Self-similar and asymptotically self-similar solutions of nonlinear wave equations*, Math. Ann. **316** (2) (2000), 259–281.

[185] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Press, Durham, NC (1976).

[186] F. Planchon, *Solutions globales et comportement asymptotique pour les equations de Navier–Stokes*, Doctoral Thesis, Ecole Polytechnique, France (1996).

[187] F. Planchon, *Global strong solutions in Sobolev or Lebesgue spaces for the incompressible Navier–Stokes equations in $\mathbb{R}^3$*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 319–336.

[188] F. Planchon, *Convergence de solutions des équations de Navier–Stokes vers des solutions auto-similaires*, Exposé n. III, Séminaire X-EDP, Ecole Polytechnique (1996).

[189] F. Planchon, *Asymptotic behavior of global solutions to the Navier–Stokes equations in $\mathbb{R}^3$*, Rev. Mat. Iberoamericana **14** (1) (1998), 71–93.

[190] F. Planchon, *Solutions auto-similaires et espaces de données initiales pour l'équation de Schrödinger*, C. R. Acad. Sci. Paris **328** (1999), 1157–1162.

[191] F. Planchon, *Self-similar solutions and Besov spaces for semi-linear Schrödinger and wave equations*, Journées "Equations aux Dérivées Partielles", Saint-Jean-de-Monts, 1999, Exp. No. IX, 11 pp., Univ. Nantes, Nantes (1999).

[192] F. Planchon, *On the Cauchy problem in Besov space for a non-linear Schrödinger equation*, Commun. Contemp. Math. **2** (2) (2000), 243–254.

[193] F. Planchon, *Sur un inégalité de type Poincaré*, C. R. Acad. Sci. Paris Sér. I Math. **330** (1) (2000), 21–23.

[194] F. Planchon, *An extension of the Beale–Kato–Majda criterion for the Euler equations*, Comm. Math. Phys. **232** (2) (2003), 319–326.

[195] G. Ponce, R. Racke, T.C. Sideris and E.S. Titi, *Global Stability of Large Solutions to the 3D Navier–Stokes equations*, Comm. Math. Phys. **159** (1994), 329–341.

[196] F. Ribaud, *Analyse de Littlewood–Paley pour la résolution d'equations paraboliques semi-linéaires*, Doctoral Thesis, Université de Paris Sud, France (1996).

[197] F. Ribaud, *Problème de Cauchy pour les équations paraboliques semi-linéaires avec données dans $H^s_p(\mathbb{R}^n)$*, C. R. Acad. Sci. Paris **322** (1) (1996), 25–30.

[198] F. Ribaud, *Semilinear parabolic equations with distributions as initial data*, Discrete Contin. Dyn. Syst. **3** (3) (1997), 305–316.

[199] F. Ribaud, *Cauchy problem for semilinear parabolic equations with initial data in $H^s_p(\mathbb{R}^n)$ spaces*, Rev. Mat. Iberoamericana **14** (1) (1998), 1–46.

[200] F. Ribaud and A. Youssfi, *Regular and self-similar solutions of nonlinear Schrödinger equations*, J. Math. Pures Appl. **77** (10) (1998), 1065–1079.

[201] F. Ribaud and A. Youssfi, *Solutions globales et solutions auto-similaires de l'équation des ondes non linéaire*, C. R. Acad. Sci. Paris **329** (1) (1999), 33–36.

[202] F. Ribaud and A. Youssfi, *Global solutions and self-similar solutions of semilinear wave equation*, Math. Z. **239** (2) (2002), 231–262.

[203] G. Rosen, *Navier–Stokes initial value problem for boundary-free incompressible fluid flow*, J. Phys. Fluid **13** (1970), 2891–2903.

[204] O. Sawada, *On time-local solvability of the Navier–Stokes equations in Besov spaces*, Adv. Differential Equations **8** (4) (2003), 385–412.

[205] V. Scheffer, *A solution to the Navier–Stokes inequality with an internal singularity*, Comm. Math. Phys. **101** (1) (1985), 47–85.

[206] M.E. Schonbek, *L^2 decay for weak solutions to the Navier–Stokes equations*, Arch. Ration. Mech. Anal. **88** (1985), 209–222.

[207] P. Serfati, *Une preuve directe d'existence globale des vortex patches 2D*, C. R. Acad. Sci. Paris Sér. I Math. **318** (6) (1994), 515–518.

[208] J. Serrin, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Arch. Ration. Mech. Anal. **9** (1962), 187–195.

[209] J. Serrin, *The initial value problem for the Navier–Stokes equations*, 1963 Nonlinear Problems, Proc. Sympos., Madison, WI, Univ. Wisconsin Press, Madison, WI (1963), 69–98.

[210] Y. Shibata, *Global solutions of nonlinear evolution equations and their stability* (in Japanese), Sūgaku **51** (1999), 1–17.

[211] M. Shinbrot, *Lectures on Fluid Mechanics*, Gordon and Breach, New York–London–Paris (1973).

[212] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20** (2) (1998), 7–15.

[213] H. Sohr, *The Navier–Stokes Equations*, *An Elementary Functional Analytic Approach*, Birkhäuser, Basel–Boston–Berlin (2001).

[214] E.M. Stein, *Harmonic Analysis*, Princeton Math. Ser., Vol. 43, Princeton Univ. Press (1993).

[215] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press (1971).

[216] M. Taylor, *Analysis on Morrey spaces and applications to the Navier–Stokes and other evolution equations*, Comm. Partial Differential Equations **17** (1992), 1407–1456.

[217] R. Temam, *Some developments on Navier–Stokes equations in the second half of the 20th century*, Development of Mathematics 1950–2000, J.-P. Pier, ed., Birkhäuser, Basel (2000), 1049–1106.

[218] R. Temam, *Navier–Stokes Equations*, *Theory and Numerical Analysis*, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI (2001).

[219] E. Terraneo, *Applications de certains espaces fonctionnels de l'analyse harmonique réelle aux equations de Navier–Stokes et de la chaleur non linéaire*, Doctoral Thesis, Université d'Evry Val d'Essonne, France (1999).

[220] E. Terraneo, *Sur la non-unicité des solutions faibles de l'équation de la chaleur non linéaire avec non-linéarité u^3*, C. R. Acad. Sci. Paris Sér. I Math. **328** (9) (1999), 759–762.

[221] G. Tian and Z. Xin, *One-point singular solutions to the Navier–Stokes equations*, Topol. Methods Nonlinear Anal. **11** (1) (1998), 135–145.

[222] S. Tourville, *An analysis of numerical methods for solving the two-dimensional Navier–Stokes equations*, Ph.D. Thesis, Washington University of St. Louis, MO (1998).

[223] S. Tourville, *Existence and uniqueness of solutions for a modified Navier–Stokes equation in $\mathbb{R}^2$*, Comm. Partial Differential Equations **23** (1–2) (1998), 97–121.

[224] S. Tourville, *Existence and uniqueness of solutions for the Navier–Stokes equations with hyperdissipation*, J. Math. Anal. Appl. **281** (1) (2003), 62–75.

[225] H. Triebel, *Theory of Function Spaces*, Monograph in Mathematics, Vol. 78, Birkhäuser, Basel (1983).

[226] H. Triebel, *Theory of Function Spaces, II*, Monograph in Mathematics, Vol. 84, Birkhäuser, Basel (1992).

[227] T.P. Tsai, *On Leray's self-similar solutions of the Navier–Stokes equations satisfying local energy estimates*, Arch. Ration. Mech. Anal. **143** (1) (1998), 29–51; Erratum, ibid. **147** (1999), 363.

[228] H.B. Veiga and P. Secchi, L^p-*Stability for the strong solutions of the Navier–Stokes equations in the whole space*, Arch. Ration. Mech. Anal. **98** (1987), 65–70.

[229] M. Vishik, *Hydrodynamics in Besov Spaces*, Arch. Ration. Mech. Anal. **145** (1998), 197–214.

[230] M. Vishik, *Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type*, Ann. Sci. Ecole Norm. Sup. **32** (6) (1999), 769–812.

[231] M. Vishik, *Incompressible flows of an ideal fluid with unbounded vorticity*, Comm. Math. Phys. **213** (3) (2000), 697–731.

[232] W. von Wahl, *The Equations of Navier–Stokes and Abstract Parabolic Equations*, Aspect der Mathematik, Vieweg und Sohn, Wiesbaden (1985).

[233] K.A. Voss, *Self-similar solutions of the Navier–Stokes equations*, Ph.D. Thesis, Yale University (1996).

[234] F.B. Weissler, *The Navier–Stokes initial value problem in* L^p, Arch. Ration. Mech. Anal. **74** (1981), 219–230.

[235] M. Wiegner, *The Navier–Stokes equations – A neverending challenge?* Jahresber. Deutsch. Math.-Verein. **101** (1999), 1–25.

[236] M. Yamazaki, *The Navier–Stokes equations in the weak-L^n spaces with time-dependent external force*, Math. Ann. **317** (2000), 635–675.

[237] M. Yamazaki, *Shuju no kansūkūkan ni okeru Navier–Stokes houteishiki* (*Navier–Stokes equations in various function spaces*) (in Japanese), Sūgaku **51** (3) (1999), 291–308; English transl.: Amer. Math. Soc. Transl. Ser. 2, Vol. 204, Selected Papers on Classical Analysis, K. Nomizu, ed., Amer. Math. Soc., Providence, RI (2001), 111–132.

[238] K. Yosida, *Functional Analysis*, Reprint of the sixth edition (1980), Classics in Mathematics, Springer-Verlag, Berlin (1995).

[239] V.I. Yudovich, *The Linearization Method in Hydrodynamical Stability Theory*, Transl. Math. Monogr., Vol. 74, Amer. Math. Soc., Providence, RI (1984).

[240] A. Zhioua, *Solutions mild pour le problème de Navier–Stokes dans le demi-espace pour une donnée singulière*, Doctoral Thesis, Université d'Evry Val d'Essonne, France (2003).