A geometric $\boldsymbol{\kappa}$-Krein space $H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$ based mechanical and dynamic quanta energy field model and related vacuum-, plasma- and Maxwell-Newton-Mie κ-quanta schema

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Dedicated to my son Mario
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Abstract:

A κ-Krein spaces based unified mathematical quanta field theory is provided. From the Mie theory the concept of discrete energy node elements (accompanied by a correspondingly defined kinetic energy Hilbert space) is taken providing an appropriate modelling framework for a physical problem specific (self-adjoint) kinetic energy operator. From the correspondingly defined extended Krein space framework the concept of a (self-adjoint) potential energy operator is applied. It enables the definition of related κ-potential energy norms on all of the κ-Krein spaces built on sets of related κ-quantum number systems.

The proposed two complementary mechanical and dynamical energy field models are accompanied by four groups of mechanical and dynamical quanta. Those mechanical and dynamical quanta are appropriately composed by two fundamental mathematical entities, an electrino and a positrino, governed by appropriately chosen sets of quantum numbers. The three mechanical relevant substances are $atom^{+}$, $atom^{-}$, and $plasma^{\pm} - pair$ particles.

The kinetical „energetical physical world“ is compactly embedded into a kinetical & potential „energetical physical world“, which is compactly embedded into an overall „energetical physical & mathematical world“ in the form $E_{tot} \coloneqq E_{kin} \otimes E_{pot} \otimes E_{gst}$. The construction of those energy Hilbert spaces is purely based on mathematical axioms and theories (e.g., number theory and Krein space theory). The corresponding (potential energy type specific) energy quanta are appropriately composed by the two fundamental mathematical entities. The different compactly embeddedness cases enable the application of the least action principle defining a best possible physical approximation solution to a solution of a mathematical variation equation model of an overall „conservation of energy“ law. The well defined least action relevant potential differences are supposed to replace the Newton resp. the Coulomb potential.

Analog to the „physical-mathematical“ embeddedness framework the least action principle can be applied to define approximation solutions of a „biological or chemical world“ (atoms & molecules, viruses & cells) to related (potential function) solutions of a „physical world“.

1. Introduction

About 95% of the universe is about the phenomenon „vacuum“. The same proportion applies to the emptyness between a proton and an electron. The remaining 5% of universe's vacuum consists roughly of 5% matter, of 25% sophisticated „*dark matter*“, and of 70% sophisticated „*dark energy*“. Nearly all (about 99%) of the 5% matter in the universe is in "plasma state". A presumed physical concept of „*dark matter*“ „explains“ the phenomenon of the spiral shapes in the universe. A presumed physical concept of „*dark energy*“ explains the phenomenon of the cosmic microwave background.

The SMEP provides a collection of assumed physical elementary particles grouped according to the three physical forces phenomena, the electromagnetic force, the weak and the strong EP interaction forces. This „*interacting*“ is governed by three independent mathematical (conservation of energy) symmetry groups governing three groups of independent fermions accompanied by corresponding „interacting“ energy field boson elements. The specification attributes of those three independent „standard elementary particle models“ are constructed according to the fundamental concept of Dirac's theory of radiation:

> *E. Fermi: „Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field“.*

Classical physics does not known about magnetism. F. Ehrenhaft's discovered phenomenon, (which he called „photophoresis“), where the light induces not only electric but also magnetic charges (poles) upon the particles if they are illuminated by concentrated light preponderantly shorter wave lengths, is still neglected, (EhF) p. 243.

Current mathematical models in plasma physics (*), distinguish between linear and nonlinear Landau damping terms (while in both cases the energetic root cause of the Landau damping phenomenon is based on the Coulomb potential), indicating that this phenomenon arises from two different physical effects, (**). At the same time, analogs to Landau damping exist in the interactions of stars in a galaxy at the Lindblad resonances of a spiral downsity wave.

A mathematical model of an unified quanta energy field is provided. The building principle of the fundamental quanta entities is motivated by Schnirelmann's density concept for subsets of integers, which is a basic concept in addititive number theory, (NaM). The set of primes excluding the integer „2“ are all odd integers. It builds the foundation of the whole number theory. The Schnirelmann density of the set of odd integers is ½, while the Schnirelmann density of the set of even integers is zero.

(*) (TsV) p. 4.: „*Plasma is that state of matter in which the atoms or molecules are found in an ionized state. The interactions of electrons and ions are determined by long-range electrical forces. The many forms of collective motion in a plasma are the result of coupling the charged-particle motion to the electromagnetic field. Therefore, the electromagnetic field which accompanies the particle motion is also a random nonreproducible quantity in a turbulent plasma. Measurements have shown that the fields excited in a plasma during the development of turbulence do in fact have a random nature*“

(CaF) p. 390 ff.: "*The turbulence of plasma differs from the hydrodynamic turbulence by the action of the magnetic field. A more relevant difference is due to the hydrodynamic interaction between the plasma particles, the interaction with the magnetic fields, and the interaction between the electromagnetic waves. ... All of them are the root cause of electromagnetic plasma turbulence. ... The case of interactions between quasi-stationary electromagnetic waves is called weak turbulence. ... The case of non-linear Landau damping (strong plasma turbulence) leads to the generation of virtual waves, which transfer their energy to the affected particles asymptotically with* $1/t$*; the plasma is heated (turbulence heating) faster than this may happen by purely particles collisions*"

(**) (ChF) p. 248-249: „*There are actually two kinds of Landau damping: linear Landau damping, and nonlinear Landau damping. Both kinds are independent of dissipative collisional mechanisms. If a particle is caught in the potential well of a wave, the phenomenon is called „trapping“. Particles can indeed gain or lose energy in trapping. However, trapping does not lie within the purview of the linear theory. …. Trapping is not in the linear theory. When a wave grows to a larger amplitude, collisonless damping with trapping occur. One then finds that the wave does not decay monotonically; rather the amplitutes fluctuates during the decay as the trapped particles bounce back and forth in the potential wells. This is nonlinear Landau damping. .. Since the linear Landau damping is derived from a linear theory, … the nonlinear Landau damping must arise from a different physical effect. The question is: Can untrapped electrons moving close to the phase velocity of the wave exchange energy with the wave*?“

In the proposed mathematical κ-Krein space based quanta potential field model the potential operator becomes an intrinsic part of the Krein space framework and not a physical phenomenon specific appropriately chosen function, like the Coulomb potential function in Dirac's theory of radiation. The (complementary) kinetical energy system of this model is defined by the energy knots of a physical phenomenon specific kinetic energy operator. The energy knots may be interpreted as the mass of the corresponding quantum element. Those „mass" quanta potential elements are composed by two mathematical quanta potential elements, the electrino and the positrino.

The conceptual new elements of the proposed mathematical model are

1. four groups of mechanical and dynamical quanta
accompanied by appropriately defined sets of quantum numbers, being composed by two fundamental mathematical entities, an electrino and a positrino:

Modelling case	mechanical quanta	dynamical quanta
Vacuum energy field	neutron: $\underline{n} = 2\nu = \epsilon\epsilon\pi\pi$	(neutrino,neutrino): $(\epsilon\pi, \epsilon\pi)$
Plasma energy field	(electron,positron): $(e = \epsilon\epsilon, p = \pi\pi)$	-
Maxwell-Mie energy field	electroton: $\underline{e} = \epsilon\epsilon\pi$	positrino π
Newton-Mie energy field	magneton: $\underline{m} = \pi\pi\epsilon$	electrino ϵ

two mathematical baseline energy field quanta types, the electrino and the positrino, which may be interpreted as

- binary quanta information carriers
Taking into account a concept of non-mechanical binary quanta information carriers the synapses (neuronal net) model is no longer restricted to mechanical signals with velocities limited by the speed of light, but also enabling other kinds of potential differences between biological synapses governed by dynamical energy quanta.

- Leibniz's concepts of (otherworldly) monades enabling preestablished (mechanical) harmony. The „binary quanta" interpretation puts the spot on related „mind & matter" resp. „mind & cosmos" topics, (**), e.g., the „philosophy of time", (CaC), especially regarding the „problem of time" with respect to „physical time" (Einstein accomapnied by multiple other physicists) vs. „duration" (Bergson), accompanied by people like Husserl, Heidegger and others, (CaJ1).

2. two complementary mechanical and dynamical energy field models

Within the calculus of variations we shall deal with κ-depending (energy) Hilbert-Krein space compositions in the form $E_{tot} := E_{mec} \otimes E_{dyn}$, where E_{mec} denotes the current mechanical energy concept of physics (which is about kinetical emergy Hilbert spaces accompanied by case specifically defined potential functions). In the proposed framework the total energy of the system is accompanied by a Hamiltonian (selfadjoint) operator H expressed as the sum of mechanical and a dynamical potential energy operators. The general concept is about a decomposition in the form $E_{tot} := E_{kin} \otimes E_{pot} \otimes E_{gst}$, where $E_{kin} = H_1$ denotes the standard energy Hilbert space in the calculus of variations accompanied by the Dirichlet integral based inner product and where E_{gst} denote the ground state energy Hilbert space. As both energy space are compactly embedded into the overall energy space E_{tot} their related energy quanta of E_{kin} and E_{pot} are governed by related least action principles in combination with a conservation of (total) energy law.

$$E_{mec} = E_{kin} \text{ and } E_{dyn} = E_{pot} \otimes E_{gst}, \text{ where } E_{kin} = H_1.$$

We note that this „decomposition" concept is in line with the Planck's considerations of dynamical and statistical types of physical laws, (PlM), which is also in line with Schrödinger's conceptions of order-from-order and order-from-disorder mechanisms in the different natural sciences, (ScE). The model is also in line with the newly proposed concept of an electric pressure of the Mie theory to explain, „why the field possesses a granular structure and why the knots of energy remain intact in spite of the back and forth flux of energy and momentum", (WeH) p. 171.

It turns out that the simplest energy operator for the total energy in a one-component system is accompanied by the domain $H_{1/2}$ equipped with a norm in the form

$$\|x\|_{1/2}^2 = \int_0^\infty \|x\|_{1.(\tau)}^2 d\tau = \sum_{n=1}^\infty \sqrt{\lambda_n} x_n^2 .$$

Therefore, the simplest $E_{tot} := E_{mec} \otimes E_{dyn}$ representation is given by $H_{1/2} = H_1 \otimes H_1^\perp$, where the related inner product of the Hilbert space H_1 is given by the Dirichlet integral in the form $D(u,v) = (\nabla u, \nabla v) = (u,v)_1$ with $\nabla : H_1 \to H_0$.

In (BrK6) an alternative Schrödinger (-Caldeón) operator is proposed, which is based on an operator in the form $i\nabla R : H_1^\perp \to H_0^\perp$, where R denotes the Riesz transform operator. In other words, the elements of the sub-space $H_1^\perp$ of $H_{1/2}$ become the alternative (energy space) quanta model replacing physical case specific potential functions of current models (which anyway only govern potential differences between virtual distances of two physical particles in space).

1. First applications of the proposed model

The κ -quanta schema of the next section are related to each other by the balances

a. $e+p \leftrightarrow \underline{n}$

b. $2\underline{m} \leftrightarrow p+\underline{n}$

c. $2\underline{e} \leftrightarrow e+\underline{n}$

d. $2\underline{m}+\epsilon \leftrightarrow \underline{e}+p+\nu$

e. $2\underline{e}+\pi \leftrightarrow \underline{m}+e+\nu$

Therefore,

3. two magnetons $\underline{m} \otimes \underline{m}$ may define one nucleon, which is in line with proton decays $2\underline{m} \rightarrow 2p+e, \underline{m} \rightarrow p+\epsilon$, while n nucleons may be interpreted as an atomic nucleus with atomic number n
4. two electrotons may define one electricon, which is in line with a $spin(1/2)$ property, while n electricons may be interpreted as n-valent ions.

The variational equations are governed by a bilinear form $a(\cdot,\cdot)\colon E_{\kappa_1} \times E_{\kappa_2} \rightarrow R$, where the parameter κ is selected according the following mechanical relevant „matter particles“ schema:

Plasma „matter“ case	$plasma^{\pm;\mp}$	$(p \otimes \mathrm{e}) \times (e \otimes \mathrm{p})$
Atom „matter“ case	$atom^{\pm;\pm}$	$(\underline{m} \otimes \underline{m}) \times (\epsilon \otimes \epsilon)$, i.e., $(atom^{+} \times e)$
Electricity „matter“ case	$atom^{\mp;\mp}$	$(\underline{e} \otimes \underline{e}) \times (\pi \otimes \pi)$, i.e. $(atom^{-} \times p)$

Putting $N^{+} := (\underline{m},\epsilon)$ and $N^{-} := (\underline{e},\pi)$ the three status cases of „plasma matter“ (cold, „medium“, hot) are modelled by the following two-component-particle schema:

Ionization of … $(atom^{\pm}, atom^{\mp}),(atom^{\mp}, atom^{\pm})$	Ionization“ percentage	two-component mechanical quanta pair	Two-component dynamical quanta pair
$(N^{+},N^{-}),(N^{-},N^{+})$	0% (cold plasma)	$(\underline{m},\underline{e})$	~~(ϵ,π), to be neglegted~~
$(N^{+},N^{-}),(N^{-},N^{+})$	100% (hot plasma)		$(e,p)=2(\epsilon\epsilon,\pi\pi)$
„medium“	$\alpha \cdot$ *#cold+β · #hot*	$\alpha \cdot \#(\underline{m},\underline{e})$	$\beta \cdot \#(e,p)$

The building principle for decompositions in the form $E_{tot} := E_{kin} \otimes E_{pot} \otimes E_{gst}$ is based on the underlying self-adjoint & positive kinematical operator. For a one-component plasma model the non-linear collision operator of the Landau equation is given by

$$Q(f,f)=\frac{\partial}{\partial v_i}\left\{\int_{R^N} a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j}-f(v)\frac{\partial f(w)}{\partial w_j}\right]dw\right\}$$

with

$$a_{ij}(z):=\frac{1}{|z|}\left\{\delta_{ij}-\frac{z_i z_j}{|z|^2}\right\}:=\frac{1}{|z|}P(z)=\frac{1}{|z|}[Id-\bar{Q}](z) \text{ and } \bar{Q}(z):=(R_i R_j)_{1\le i,j\le N},$$

where $P(z)$ resp. R_i denote the Leray-Hopf resp. Riesz operators, and where $a(z)$ is symmetric, non-negative and even in z; f denotes an unknown function corresponding at each time t to the density of particle at the point x with velocity v.

The Leray-Hopf operator with the symbol $b_{ij}(z)=z a_{ij}(z)$ may be interpreted as a kind of linerized Landau operator. It is of order zero. In (LeN) the action of the Leray-Hopf operator on Gaussian functions is provided, accompanied by the standard Laplace operator and the Oseen matrix operator. In (WeP), (WeP1) self-adjoint extensions of the Laplacian operator with respect to electric and magnetic boundary conditions are provided. The Leray-Hopf operator is proposed as plasma collision model operator accompanied by appropriate eigenpair solutions enabling the definition of corresponding Hilbert scales, (BrK5).

2. The mathematical model

a. The $\boldsymbol{\kappa}$-Krein space framework $\boldsymbol{H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}}$

Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. The Hilbert spaces $\{H_\alpha | \alpha \in R\}$ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \, , \, x_n := (x, \varphi_n)$$

accompanied by the inner product $(x,y)_\alpha = \sum_1^\infty \lambda_n^\alpha \, x_n \, y_n$.

In case of $\alpha = 0$ we skip the subscript. The physical model problem for the operator A is the Friedrichs extension of the Laplacian operator $A := \overline{-\Delta\|\quad\|_1}$ with domain $D(A) = H_1$. Then, the bilinear form $a(u,v) := (Au, v)$ defines an inner (kinetic energy) product in $D(A) = H_1$ and the operator equation $-\Delta u = f$ is equivalent to the weak (variational) representation in the form, (BrK),

$$(u,v)_1 = (f,v), \forall v \in H_1.$$

For $\alpha < 0$ the Fourier coefficients x_n contribute to the α -norm with a polynomial decay. The extended Hilbert space $H_{(\tau)}$ is defined by the inner product resp. norm

$$(x,y)_{(\tau)} = \sum_1^\infty e^{-\sqrt{\lambda_i}\tau} \, x_n \, y_n, \quad \|x\|_{(\tau)}^2 = (x,x)_{(\tau)}.$$

The (τ)-norm is weaker than any α-norm, i.e.

$$\|x\|_{(\tau)}^2 \le c\|x\|_\alpha^2 \quad \text{for any } \alpha\text{-norm}$$

with $c = c(\alpha, \tau)$ depending only on α and τ.

Let $\Phi_n := \varphi_n^H$ denote the Hilbert transform of φ_n with $(\varphi_n, \Phi_n) = 0$, (*), (BrK1). Then, the system $\left\{\psi_{n.\tau}^{(1)}, \psi_{n.\tau}^{(2)}\right\}$ with

$$\psi_{n.\tau}^{(1)} := \varphi_n - i\Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} \, , \quad \psi_{n.\tau}^{(2)} := \varphi_n + i\Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}$$

defines an orthogonal system of the Hilbert space composition $H_0 \otimes H_{(\tau)}$.

In quantum mechanics the total energy of a system is given by a Hamiltonian (selfadjoint) operator H expressed as the sum of a kinetic and a potential energy operator in the form $H_{mech} = H_{kin} + E_{pot}$.

The conceptual design of the proposed integrated mathematical model is based on a Hermitian operator expressed as the sum of two hermitian mechanical and dynamical operators. The domain of the mechanical energy operator is accompanied by the (weak) standard domain H_1. The domain of the dynamical energy operator is accompanied by domains like $H_{1/2}$ and $H_{1/2.\kappa}$ equipped with the norms like

$$\|x\|_{1/2}^2 = \int_0^\infty \|x\|_{1.(\tau)}^2 d\tau = \sum_{n=1}^\infty \sqrt{\lambda_n} x_n^2$$

$$|||x|||_{1/2.\kappa}^2 = \sum_{n=1}^\infty \sqrt{\lambda_n} x_n^2 \int_0^\infty tanh^2(\kappa_n \tau) \, e^{-\sqrt{\lambda_n}\tau} d\tau \, , \, \kappa_n \neq 0.$$

The related sequences

$$\kappa_{\tau.n}^{+} := \frac{1}{2} \frac{e^{\kappa_n \tau}}{\cosh(\kappa_n \tau)} \, , \, \kappa_n^{-} := \frac{1}{2} \frac{e^{-\kappa_n \tau}}{\cosh(\kappa_n \tau)} \quad \text{with } \kappa_n \in R$$

define a Krein space decomposition of the Hilbert space $H_{(\tau)}$ in the form $H_{(\tau)} = H_{\kappa.(\tau)}^{+} \otimes H_{\kappa.(\tau)}^{-}$.

(*) for space dimensions greater than one the counterpart of the Hilbert transform operator is the Riesz transform operator; for a correspondingly defined alternative Schrödinger (-Calderón) momentum operator we refer (BrK5); for corresponding well-defined hybrid/mixed Ritz-Galerkin approximations we refer to (NiJ).

The Hilbert space $H_{(\tau)}$ and the Krein space $H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$ are accompanied by two related inner products on all of the Hilbert space $H_{(\tau)}$ in the form [(*)]

$$(x,y)_{(\tau)} = \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

$$((x,y))_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, x_n y_n e^{-\sqrt{\lambda_n}\tau} .$$

For $x = x_1 + x_{1/2.\kappa} \in H_1 \otimes H_{1/2.\kappa}$ and

$$(((x,y)))_{1/2.\kappa} := (x_1, y_1)_1 + \left((x_{1/2.\kappa}, y_{1/2.\kappa})\right)_{1/2.\kappa}$$

the weak (variational) representation of a Hermitian (energy) operator equation is given by

$$u \in H_1 \otimes H_{1/2.\kappa} : \quad (((u,v)))_{1/2.\kappa} = (f,v), \forall v \in H_1 \otimes H_{1/2.\kappa}.$$

The Hilbert-Krein spaces are associated with Sobolev spaces. The underlying domains are associated with the complex Lorentz group, which is associated with $SL(2,C) \otimes SL(2,C) \cong SU(2) \otimes SU(2) \cong S^3 \otimes S^3$. The variational representation enables approximation methods in Hilbert scales (compactly embeddedness of $H_2 \subset H_1 \subset H_{1/2}$) for underlying (standard) statistical relevant kinetic and potential function solutions.

The Krein space is also equipped with the indefinite inner products resp. metric [(*)]

$$[x,y]_{\kappa.(\tau)} := \left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) - \left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right) = \sum_{n=1}^{\infty} tanh\,(\kappa_n\tau) e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

with

$$\left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) := \sum_{n=1}^{\infty} (\kappa^{+}_{\tau.n})^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n \;\;,\;\; \left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right) := \sum_{n=1}^{\infty} (\kappa^{-}_{\tau.n})^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n \;.$$

The indefinite norm $[x,x]_{\kappa.(\tau)}$ may be interpreted as a potential (functional), (VaM) p. 90,

$$\varphi_{\kappa.\tau}(x) := [x,x]_{\kappa.(\tau)} = \left\|x^{+}_{\kappa.(\tau)}\right\|^2 - \left\|x^{-}_{\kappa(\tau)}\right\|^2 = \sum_{n=1}^{\infty} \tanh(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau} x_n^2 \;.$$

The self-adjoint operator

$$W_{\kappa.\tau} x := x^{+}_{\kappa.(\tau)} - x^{-}_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh(\kappa_n\tau) e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n$$

may be interpreted as the quantum potential operator of the considered κ -quantum potential energy systems.

The definition of the potential operator enables a treatment of the results of its action as the „mirror reflection“ of the space $H_{(\tau)}$ in the subspace $H^{+}_{\kappa.(\tau)}$. The sub-space $H^{+}_{\kappa.(\tau)}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = 1$. The sub-space $H^{-}_{\kappa.(\tau)}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = -1$. The whole spectrum of $W_{\kappa.\tau}$ lies on the join of the points $\lambda = \pm 1$.

From the equivalent formulas

$$(x,y)_{(\tau)} = \left[x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right] - \left[x^{-}_{\kappa.(\tau)}, y^{-}_{\kappa.(\tau)}\right]$$

$$[x,y]_{\kappa.(\tau)} := \left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) - \left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right)$$

it follows the characterization of „positive“, „negative“, and „neutral“ vectors $x \in H_{(\tau)}$ by the relations

$$\left\|x^{+}_{\kappa.(\tau)}\right\| > \left\|x^{-}_{\kappa(\tau)}\right\| \;,\; \left\|x^{+}_{\kappa.(\tau)}\right\| < \left\|x^{-}_{\kappa.(\tau)}\right\| \;,\; \left\|x^{+}_{\kappa.(\tau)}\right\| > \left\|x^{-}_{\kappa(\tau)}\right\| .$$

[(*)] The relation to the proposed potential energy norms is given by the equality $\|x\|^2_{1/2} = \int_0^{\infty} \|x\|^2_{1.(\tau)} d\tau = \sum_{n=1}^{\infty} \sqrt{\lambda_n} x_n^2$.

Putting $x_{(\tau)} := \sum_{n=1}^{\infty} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H_{(\tau)}$, $x^{+}_{\kappa.(\tau)} := \sum_{n=1}^{\infty} \kappa^{+}_{\tau.n} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H^{+}_{\kappa.(\tau)}$, $x^{-}_{\kappa.(\tau)} := \sum_{n=1}^{\infty} \kappa^{-}_{\tau.n} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H^{-}_{\kappa.(\tau)}$

it follows $\kappa^{+}_{\tau.n} + \kappa^{-}_{\tau.n} = 1$, $\kappa^{+}_{\tau.n} - \kappa^{-}_{\tau.n} = \tanh(\kappa_n\tau)$, $(\kappa^{+}_{\tau.n})^2 - (\kappa^{-}_{\tau.n})^2 = \frac{sinh\,(2\kappa_n\tau)}{cosh^2(\kappa_n\tau)} = \tanh(\kappa_n\tau)$

$$[x,y]_{\kappa.(\tau)} := \left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) - \left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{sinh\,(2\kappa_n\tau)}{cosh^2(\kappa_n\tau)} e^{-\sqrt{\lambda_n}\tau} x_n y_n = \sum_{n=1}^{\infty} tanh\,(\kappa_n\tau) e^{-\sqrt{\lambda_n}\tau} x_n y_n \;.$$

The potential $\varphi_{\kappa.\tau}(x)$ in combination with the functional $((x)) := \sqrt{\varphi_{\kappa.\tau}(x)}$ generates hyperboloids H_c, hyperbolic regions V_c, and conical regions V_0 in the form

$$H_c := \{x \in H_{(\tau)} | \varphi_{\kappa.\tau}(x) = c > 0\}, V_c := \{x \in H_{(\tau)} | ((x)) \geq c > 0\}, V_0 := \{x \in H_{(\tau)} | ((x)) \geq 0\}.$$

Evidently V_c is a subspace of V_0. The boundary K of the conical region is defined by the condition $((x)) = 0$. It is an asymptotic conical manifold for the hyperboloid $((x)) = c > 0$ (*).

The counterparts of the W-norms $|||x|||^2_{\kappa.(\tau)} := [W_{\kappa.\tau}x, x]_{\kappa.(\tau)}$ with respect to the H_α Hilbert space norms $\|x\|^2_\alpha = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty$ are given by the norms

$$|||x|||^2_{\alpha.\kappa} := \sum_{n=1}^\infty tanh^2(\kappa_n \tau)\, \lambda_n^\alpha x_n^2 .$$

Let $L := H_{\alpha.\kappa} \subset H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$ and $P^\pm$ be the canonical projectors. Then the set of vectors of L can be represented in the form

$$L := H_{\alpha.\kappa} := \{x^+_{\alpha.\kappa} + K^+ x^+_{\alpha.\kappa}\}_{x^+ \in H^+_{\kappa.\alpha}}$$

giving the general form of all $H^+_{\kappa.\alpha} \subset H^+_{\kappa.(\tau)}$ of the Krein space $H = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$. The bounded linear operator

$$K^+ = K^+_{\kappa.(\tau)} := P^-(P^+|H_{\alpha.\kappa})^{-1} : P^+|H_{\alpha.\kappa} \rightarrow H^-_{\kappa.(\tau)}$$

is called the angular operator for $H_{\alpha.\kappa}$ with respect to $H^+_{\kappa.(\tau)}$ (**). The inclusion $H^+_{\kappa.\alpha} \subset H^+_{\kappa.(\tau)}$ is accompanied by related inclusions $H^-_{\kappa.\alpha} \subset H^-_{\kappa.(\tau)}$. The related Krein space concept is called alternating (maximal) pairs and alternating extensions, (***). This concept can be applied in the context of dissipative operators in Hilbert spaces, (BoJ) p. 116.

(*) (VaM) p. 91: „*If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0,\infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $((x)) = c > 0$ in a single point.*

We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $((x)) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\|x^+_{\kappa.(\tau)}\| = \|x^-_{\kappa(\tau)}\|$ belong to K, and those points of S^1 for which $\|x^+_{\kappa.(\tau)}\| > \|x^-_{\kappa(\tau)}\|$ intersect the hyperboloid $((x)) = c > 0$ at the point whose distance from θ is given by $t = c(\|x^+_{\kappa.(\tau)}\|^2 - \|x^-_{\kappa(\tau)}\|^2)^{-1/2}$.

From this it is seen that $t \rightarrow \infty$ if $\|x^+_{\kappa.(\tau)}\|^2 - \|x^-_{\kappa(\tau)}\|^2 \rightarrow 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $((x)) = c > 0$.“

(**) The subspace $L \subset H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$ is positive if and only if the angular operator K^+ of L with respect to $H^+_{\kappa.(\tau)}$ exists and satisfies the condition

$$|||K^+_{\kappa.(\tau)} x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} \leq |||x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)}, \; x^+_{\kappa.(\tau)} \in D(K^+_{\kappa.(\tau)}).$$

In particular, positive definite subspaces are characterized by the property

$$|||K^+_{\kappa.(\tau)} x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} < |||x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)}, \; x^+_{\kappa.(\tau)} \in D(K^+_{\kappa.(\tau)}), \; x^+_{\kappa.(\tau)} \neq 0,$$

and neutral subspaces by

$$|||K^+_{\kappa.(\tau)} x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} = |||x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)}, \; x^+_{\kappa.(\tau)} \in D(K^+_{\kappa.(\tau)}).$$

(***) The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1), D(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1 x_1, x_1] + [x_1, T_1 x_1] \leq 0, \; x_1 \in D(T_1)$$

$$[T_2 x_2, x_2] + [x_2, T_2 x_2] \leq 0, \; x_2 \in D(T_2).$$

b. The integrated vacuum, plasma, and Maxwell-Newton-Mie $\boldsymbol{\kappa}$-quanta schema

The Krein space based vaccum, plasma, and Maxwell-Newton-Mie κ-quantum potential systems are defined by related appropriately defined sets of quantum numbers κ_n according to the following table:

Model case	**EP**	**Anti-EP**	**QN** quantum numbers	**QN** quantum numbers	**QN** quantum numbers
	$q^+ \in H^+_{\kappa.(\tau)}$	$q^- \in H^-_{\kappa.(\tau)}$	q_n^+	q_n^-	$\boldsymbol{\kappa_n} := q_n^+ - q_n^-$
Vacuum particles Electrino ϵ	ϵ	$\epsilon \otimes \pi \otimes \pi$	$n_\epsilon := \frac{n-1/2}{4n-1}$	$\frac{3n-1/2}{4n-1}$	$\kappa_\epsilon = -\frac{2n}{4n-1}$
Vacuum particles Positrino π	π	$\pi \otimes \epsilon \otimes \epsilon$	$n_\pi := \frac{n}{4n-1}$	$\frac{3n-1}{4n-1}$	$\kappa_\pi = -\frac{2n-1}{4n-1}$
Vacuum particles Neutrino ν	$\nu = \epsilon \otimes \pi$	$\nu = \epsilon \otimes \pi$	$n_\nu = \frac{1}{2}$	$\frac{2n-1/2}{4n-1} = \frac{1}{2}$	$\kappa_\nu = 0$
Plasma particles Neutron $\underline{n}$	$\epsilon \otimes \epsilon \otimes \pi \otimes \pi$	$-$	$n_{\underline{n}} = \frac{4n-1}{4n-1} = 1$	0	$\kappa_{\underline{n}} = 1$
Plasma particles electron/positron $e\,/\,p$	$e := \epsilon \otimes \epsilon$	$p := \pi \otimes \pi$	$n_e = \frac{2n-1}{4n-1}$	$n_p = \frac{2n}{4n-1}$	$\kappa_e = -\frac{1}{4n-1}$
Plasma particles positron/electron $p\,/\,e$	$p := \pi \otimes \pi$	$e := \epsilon \otimes \epsilon$	$n_p = \frac{2n}{4n-1}$	$n_e = \frac{2n-1}{4n-1}$	$\kappa_p = \frac{1}{4n-1}$
Maxwell-Mie particles Electroton $\underline{e}$	$\underline{e} := e \otimes \pi$ $\underline{e} = \epsilon \otimes \epsilon \otimes \pi$	π	$n_{\underline{e}} = \frac{3n-1}{4n-1}$	$\frac{n}{4n-1}$	$\kappa_{\underline{e}} = \frac{2n-1}{4n-1}$
Newton-Mie particles Magneton $\underline{m}$	$\underline{m} := p \otimes \epsilon$ $\underline{m} = \pi \otimes \pi \otimes \epsilon$	ϵ	$\underline{m} = \frac{3n-1/2}{4n-1}$	$\frac{n-1/2}{4n-1}$	$\kappa_{\underline{m}} = \frac{2n}{4n-1}$

Mathematical balances affecting the physical entities:
plasma pair: $(\boldsymbol{e = \epsilon\epsilon, p = \pi\pi})$, neutron: $\boldsymbol{\underline{n} = 2\nu = \epsilon\epsilon\pi\pi}$,
electroton: $\boldsymbol{\underline{e} = \epsilon\epsilon\pi}$, magneton: $\boldsymbol{\underline{m} = \pi\pi\epsilon}$

$e + p \ \leftrightarrow\ \underline{n}\ ,\ 2\underline{e}\ \leftrightarrow\ e + \underline{n}\ ,\ 2\underline{m}\ \leftrightarrow\ p + \underline{n},$ where two magnetons equal one nucleon

$$2\underline{m} + \epsilon \ \leftrightarrow\ \underline{e} + p + \nu\ ,\ 2\underline{e} + \pi \ \leftrightarrow\ \underline{m} + e + \nu.$$

nucleon decay: $2\underline{m} \ \rightarrow\ 2p + e$, electricon decay: $2\underline{e} \ \rightarrow\ 2e + p$

Remark: The underlying concept of the $q_n^\pm$ quantum number domains is about a split of odd and even numbers. The experimental observations of the spectra of atoms and their decomposition into magnetic and electric fields showed a decomposition of spectral lines or of electron beams into an even number of components, while the angular momentum multiplets were only composed by an odd number of multiplets with the numbers $2l+1$, (RoH) p. 217.

Remark: The ranges of the considered sets of quantum number systems are $\kappa_\epsilon \in [-2/3, -1/2[$, $\kappa_\pi \in]-1/2, -1/3]$, $\kappa_e \in [-1/3, -1/4[$, $\kappa_p \in]1/4, 1/3]$, $\kappa_{\underline{e}} \in [1/3, 1/2[$, $\kappa_{\underline{m}} \in]1/2, 2/3]$. Beside the (neutrino-neutron) vacuum-Mie quanta pair, the physical-statistical laws relevant quanta are given by the (electron-positron) plasma-Mie quanta pair, and the (electroton-magneton) Maxwell-Newton-Mie quanta.

Remark (appendix): In the case, where the positive part of the spectrum of $W_{\kappa.\tau}$ lies in an interval $[m, b]$, where $m > 0$, then the inequality

$$\|W_{\kappa.\tau}x\|_{(\tau)} \geq \frac{m}{\sqrt{2}}\sqrt{\varphi^2_{\kappa.\tau}(x) + \|x\|^2_{(\tau)}} \geq \frac{m}{\sqrt{2}}\sqrt{c^2 + \|x\|^2_{(\tau)}}$$

holds for every x in the hyperbolic region V_c defined by $\sqrt{\varphi_{\kappa.\tau}(x)} \geq c > 0$, as well as in the conical region V_0, i.e., when $c = 0$.

3. Appendix

Supporting Mathematics

The Hilbert scale $\{\boldsymbol{H}_{\boldsymbol{\alpha}} | \boldsymbol{\alpha} \in \boldsymbol{R}\}$

(BrK1)

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$ and the norm $\|..\|$ Let A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A).$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \rightarrow \lambda_i^{-1}$ we have from the previous section

there is a countable sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_i = \lambda_i \phi_i \ , \ (\phi_i, \phi_k) = \delta_{i,k} \quad \text{and} \quad \lim_{i \to \infty} \lambda_i \rightarrow \infty$$

any $x \in H$ is represented by

$$(*) \quad x = \sum_{i=1}^{\infty} (x, \phi_i) \, \phi_i \ \text{ and } \ \|x\|^2 = \sum_1^{\infty} (x, \phi_i)^2.$$

Similarly one can define the spaces H_α, where the case $\alpha < 0$ is related to the theory of distributions. They consist of those elements $x \in H$ with scalar product

$$(x, y)_\alpha = \sum_i^{\infty} \lambda_i^\alpha (x, \phi_i) \, (y, \phi_i) = \sum_i^{\infty} \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (*) it holds

$$\|x\|_2^2 = (Ax, Ax)_0 \, , H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices, (NiJ), (NiJ1):

Lemma :

i) Let $\alpha < \beta$. Then $\|x\|_\alpha \leq \|x\|_\beta$ for $x \in H_\beta$ and the embedding $H_\beta \to H_\alpha$ is compact.

ii) Let $\alpha < \beta < \gamma$. Then $\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu$ for $x \in H_\gamma$ with $\mu = \frac{\gamma-\beta}{\gamma-\alpha}$ and $\nu = \frac{\beta-\alpha}{\gamma-\alpha}$.

iii) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

iv) $\|x - y\|_\alpha \leq t^{\beta-\alpha}\|x\|_\beta$

v) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

vi) $\|y\|_\gamma \leq t^{-(\gamma-\beta)}\|x\|_\beta$

Lemma:

i) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

ii) $\|x - y\|_\rho \leq t^{\beta-\rho}\|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

iii) $\|y\|_\sigma \leq t^{-(\sigma-\beta)}\|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Eigen-functions and Eigen-differentials

Let H be a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|\cdot\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1}compact. Then the operator $K=A^{-1}$does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \rightarrow L(H,H)$$

$$\lambda \rightarrow E_\lambda := \int_{\lambda_0}^{\lambda} \phi_\mu(\phi_\mu,*)d\mu \quad , \quad \mu \in [\lambda_0,\infty) ,$$

i.e.

$$dE_\lambda = \phi_\lambda(\phi_\lambda,*)d\lambda .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

E_λ is a projection operator for all $\lambda \in R$

for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

$\lim\limits_{\lambda\to-\infty} E_\lambda = 0$ and $\lim\limits_{\lambda\to\infty} E_\lambda = Id$

$\lim\limits_{\substack{\mu\to\lambda \\ \mu>\lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f:[a,b] \rightarrow R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \rightarrow H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) .$$

Symbolically one writes $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \ , \ d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax,y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \qquad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2x,y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ , \ \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function. It holds the following eigen-pair relations

$$A\phi_i = \lambda_i \phi_i \quad A\phi_\lambda = \lambda\phi_\lambda \quad \|\phi_\lambda\|^2 = \infty \, , (\phi_\lambda, \phi_\mu) = \delta(\phi_\lambda - \phi_\mu).$$

The ϕ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\phi_\lambda(x) = \lambda\phi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\phi_\lambda(x)$ it results into elements of H. The practical way to find eigen-differentials is looking for solutions of a distribution equation.

The extended Hilbert space $H_{\alpha.(\tau)}$

(BrK1), (NiJ), (NiJ1)

The extended Hilbert space $H_{\alpha.(\tau)}$ is defined by the following inner product resp. norm

$$(x,y)_{(\tau)} = \sum_{i=1} e^{-\sqrt{\lambda_i}\tau}(x,\phi_i)(y,\phi_i), \;\; \|x\|^2_{(\tau)} = (x,x)_{(\tau)}.$$

The (τ)-norm is weaker than any α-norm, i.e.

$$\|x\|^2_{(\tau)} \leq c\|x\|^2_{\alpha} \quad \text{for any } \alpha\text{-norm}$$

with $c = c(\alpha,\tau)$ depending only on α and τ.

The counterpart of the related lemmata of the considered Hilbert scale is

Lemma: Let $\tau,\delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_\tau(x)$ according to

$$\|x-y\| \quad \leq \|x\|$$

$$\|y\|_1 \leq \delta^{-1}\|x\|$$

$$\|x-y\|_{(\tau)} \leq e^{-\tau/\delta}\|x\|.$$

Any Hilbert scale norm with negative index, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (τ)-norm:

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|^2_{-\alpha} \leq \delta^{2\alpha}\|x\|^2_0 + e^{\tau/\delta}\|x\|^2_{(\tau)}$$

with $\delta > 0$ being arbitrary.

Proof: The inequality is a consequence of the following inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{\tau(\delta^{-1}-\sqrt{\lambda})}, \text{ for any } \tau,\delta,\alpha > 0 \text{ and } \lambda \geq 1.$$

If $\lambda^{-1/2} \leq \delta$ then obviously $\lambda^{-\alpha} \leq \delta^{2\alpha}$, in case of $\lambda^{-1/2} \geq \delta$ it holds $e^{\tau(\delta^{-1}-\sqrt{\lambda})} \geq 1$, whereas $\lambda^{-\alpha} \leq 1$ is a consequence of $\alpha > 0$ and $\lambda \geq 1$.

Putting $\delta = \frac{1}{\theta}$ and $\lambda = \vartheta^2 \geq 1$ it follows from the lemma above the

Corollary: for any $\tau,\theta,\alpha > 0$ and $\vartheta \geq 1$ the following inequality is valid

$$\vartheta^{-2\alpha} \leq \theta^{-2\alpha} + e^{\tau(\theta-\vartheta)}.$$

Lemma: Because of $\int_0^\infty e^{-\sqrt{\lambda_i}\tau}d\tau = \frac{1}{\sqrt{\lambda_i}}$ it holds

$$\int_0^\infty \|x\|^2_{(\tau)}d\tau = \|x\|^2_{-1/2}.$$

Strong elliptic and hyperbolic PDO
(BrK1)
www.navier-stokes-equations.com

By construction the Hilbert scales characterized by a polynomial decay in case of λ_i^α enables optimal shift theorem for the Laplacian operator in the form, (appendix I)

$$\|x\|_{\alpha+2}^2 = (Ax, Ax)_\alpha = \|Ax\|_\alpha^2 .$$

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert (wave) operator related to the wave equation:

$$\Box w := \ddot{w} - \Delta w \, .$$

The Hilbert space defined by the inner product resp. norm

$$(x,y)_{(t)}^2 = \sum_{i=1}^\infty e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \;\; t>0$$

$$\|x\|_{(t)}^2 = (x,x)_{(t)}^2$$

provides „optimal" shift theorems for related strong hyperbolic operators.

Theorem: For the D'Alembert (wave) operator it holds

$$\int_0^T \|w\|_{k+2,(t)}^2 dt \le c \int_0^T \|f\|_{k,(t)}^2 dt \, .$$

Proof: Let $w_i := (w, \phi_i)$ resp. $f_i := (f, \phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-w_i'' = \lambda_i w_i$ of the Laplacian operator. Th corresponding solution of ($\Box w = f$),

$$\ddot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \text{ and } w_i(0) = \dot{w}_i(0) = 0 \, .$$

is given by

$$w_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin(\sqrt{\lambda_i}(t-\tau)\, f_i(\tau) d\tau .$$

It holds for $\tau \le t$

$$\int_0^T \|w\|_{k+2,(t)}^2 dt = \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} w_i^2(t) dt$$

$$\le \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\frac{1}{\sqrt{\lambda_i}} \int_0^t \sin(\sqrt{\lambda_i}(t-\tau)\, f_i(\tau) d\tau\right]^2 dt$$

$$\le \sum \lambda_i^{k+1} \int_0^T e^{-\sqrt{\lambda_i}t} (\int_0^t \sin(\sqrt{\lambda_i}(t-\tau) d\tau) \left[\int_0^t \sin(\sqrt{\lambda_i}(t-\tau) d\tau\, f_i^2(\tau) d\tau\right] dt$$

$$\le \sum \lambda_i^{k+1/2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\int_0^t f_i^2(\tau) d\tau\right] dt \, .$$

Exchanging the order of integration gives

$$\int_0^T \int_0^t e^{-\sqrt{\lambda_i}t} f_i^2(\tau) d\tau dt = \int_0^T \int_t^T e^{-\sqrt{\lambda_i}t} f_i^2(\tau) dt d\tau$$

$$= \int_0^T f_i^2(\tau) dt \left[\int_t^T e^{-\sqrt{\lambda_i}t} d\tau\right]$$

$$\le \frac{1}{\sqrt{\lambda_i}} \int_0^T f_i^2(\tau) dt \, .$$

Theorem: In general there exists no „optimal" hyperbolic shift theorem the standard Sobolev space framework in the form

$$\|w\|_{k+2}^2 \leq c\|f\|_k^2$$

Proof: the counter example is given by the function

$$\Phi(x,t) := e^{-(\frac{1}{2}-(x-t))^2}, u(x,t) := t^2\Phi(x,t), f(x,t) := 2\Phi(x,t) - 4t\Phi'(x,t)$$

fulfilling the relationships

$$\dot{\Phi}(x,t) = -\Phi'(x,t), \ddot{\Phi}(x,t) = \Phi''(x,t), \ddot{u}(x,t) - u''(x,t) = f(x,t)$$

and

$$\|u''\|_{L_2(L_2)} \sim \|\Phi''\|_{L_2(L_2)} \quad \text{but} \quad \|f\|_{L_2(L_2)} \sim \|\Phi'\|_{L_2(L_2)}.$$

Calculus of variations: the energy method

(VeW) p. 44

Let E denote a linear space, and U a linear subspace of E. We consider the boundary value problem as operator equation in the half-homogeous form

$$Au = f, u \in U$$

with a solution $\bar{u} \in U$. Additionally we assume

i) $(Au, v) = (u, Av), \quad \forall u, v \in U$

ii) $(Au, u) > 0, \qquad \forall u \in U, u \neq 0.$

This means, that the operator $A : U \rightarrow E$ is symmetric and positive. Then it follows that $Au = 0$ in U posseses only the solution $u = 0$, i.e., $\bar{u} \in U$ becomes the unique solution of $Au = f$.

Obviously, by

$$[u, v] \coloneqq (Au, v), \; |||u||| \coloneqq (Au, u)^{1/2}$$

There is an additonal inner product defined in U accompanied by an additional corresponding norm, which is denoted as „energy norm“ (in applications this norm often represents the physical notions „work“ or „energy“) Correspondingly, the inner product $[\cdot,\cdot]$ is called energetic inner product.

We now consider the so-called energy functional

$$I(u) = (Au, u) - 2(f, u).$$

As above, $\bar{u} \in U$ denotes the solution of $Au = f, u \in U$. Then it holds for all $u \in U$ it holds

$$I(u) = |||u - \bar{u}|||^2 - |||\bar{u}|||^2.$$

For the right side it holds

$$|||u - \bar{u}|||^2 - |||\bar{u}|||^2 = |||u|||^2 - 2[\bar{u}, u] = (Au, u) - 2(f, u) = I(u).$$

From $I(u) = |||u - \bar{u}|||^2 - |||\bar{u}|||^2$ it follows that $I(\bar{u}) = -|||\bar{u}|||^2$ and $I(u) = I(\bar{u}) + |||u - \bar{u}|||^2$.

Therefore, it holds $I(u) > I(\bar{u})$ für $u \neq \bar{u}$. In summary this means

Theorem: The operator equation $Au = f$, $u \in U$, is equivalent to the extrmal problem

$$I(u) \rightarrow min, u \in U.$$

The characterization of the solution $\bar{u}$ as a solution of the extremal problem defined by the energy functional is called the energy method.

Quadratic extremal problems for linear variational equations

(VeW) p. 48

Let E denote a linear space, and U a linear subspace of E. Additionally, let $l(\cdot) : E \to R$ denote a linear functional, and $a(\cdot,\cdot) : E \times E \to R$ a bilinear form with the following properties

iii) $a(u,v) = a(v,u), \ \forall u,v \in E$

iv) $a(u,u) \geq 0, \quad \forall u \in E$

v) $a(u,u) > 0, \quad \forall u \in U, u \neq 0$

Then, by $|||u||| := a(u,u)^{1/2}$ there is a half-norm defined in E, which is a norm in U (again called energy norm).

The extremal problem: For a given $u_0 \in E$ we look for a $u \in E$ as a solution of

$$J(u) = a(u,u) - 2l(u) \ \to min, \ u - u_0 \in U.$$

In order to enable the existence of such a solution it requires additional assumptions. However, the uniqueness of such a solution is guaranteed. Besides, the extremal problem is equivalent to the variational equation in the form

$$a(u,\varphi) = l(\varphi) \ \forall \varphi \in U, \ u - u_0 \in U.$$

The generalizations for physical relevant problems (Boltzmann equations, NSE equations, ..) is accompanied by the constructio of an operator-algebra consistent of integral and differential operators, leading to the concept of pseudo-differential operators. The counterpart of the symmetric and positive linear operator (accompanied by the energy norm) is Garding's inequality for strong elliptic pseudo-differential operators. In simple words, there is no conceptually difference regarding the application of the „energy method" for nonlinear strong elliptic or strong hyperbolic pseudo-differential operators. The non-linear terms of such operators may be interpreted as compact disturbances of the linear operator, defining the energy norm.

Non-linear minimization problems

Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW):

$$(*) \quad J(u): a(u,u) - F(u) \to min\,, \quad u - u_0 \in U.$$

Let $a(\cdot,\cdot) : \ V \times V \to R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ and $F(\cdot): V \to R$ a functional with the following properties:

$F(\cdot): V \to R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \leq (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

$$F(u) \geq \alpha \text{ for every } u \in u_0 + U$$

$F(\cdot): V \to R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \to R$ with

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v).$$

Then the minimum problem (*) is equivalent to the variational equation

$$a(u,\phi) + F_u(\phi) = 0 \text{ for every } \phi \in U$$

and admits only an unique solution.

In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \to R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The Hilbert transform operator & the mean ergotic theorem

Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. Then Hilbert spaces $\{H_\alpha | \alpha \in R\}$ and H_τ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \text{ , } \|x\|_{(\tau)}^2 = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n^2,\, x_n := (x, \varphi_n).$$

The Hilbert transform of the orthogonal system $\Phi_n := \varphi_n^H := H[\varphi_n]$, where $(\Phi_n, \varphi_n) = 0$ provides an unitary operator U on those Hilbert spaces and theory Hilbert sub-space.

Mean ergotic theorem (HaP), (HoE): Let U be an isometry on a Hilbert space H; let P be the projection on the space of all x invariant under U, then

$$\frac{1}{n}\sum_{j=0}^{n-1} U^j x \to Px \text{ in a weak } L_2 \text{ sense for all } x \in H.$$

Note: If $x = y - Uy$ for some y, then $\frac{1}{n}\sum_{j=0}^{n-1} U^j x$ is a telescoping sum equal to $y - U^n y$ and $\left\|\frac{1}{n}\sum_{j=0}^{n-1} U^j x\right\| \le \frac{2}{n}\|y\| \to 0$.

Quadratic and complementary „least energy“ Riesz-Galerkin methods

Hilbert-Krein space based least energy variational pseudo-differential equation representations enable the full power of quadratic and complementary „least energy“ Riesz-Galerkin methods accompanied by FEM, BEM, and wavelet approximation methods, (BrK).

In (NiJ2) an extension of the stndard „inf-sup-condition“ in the FEM is provided, in case applications where the underlying Banach spaces coincide and ar the cartesian product of two Hilbert spaces $X = Y = H \times H$.

The construction of an operator algebra consisting of integral and differential operators leads to the concept of pseudo-differential operators. The PDO theory provides the appropriate framework for affected physical differential and (singular) integral equations. In order to apply „least energy“ Riesz-Galerkin methods it requires strong elliptic pseudo-differential operators, (BrK1). The hyperbolic wave equation operator (the D'Alembert operator) with domain in a $H_{(\tau)}$ framework defines a strong hyperbolic pseudo-differential operators (BrK1). This allows to revisit the current concept of „wave front sets“ of the standard pseudo-differential operator theory, (PeB).

Complementary variational principles and the method of Noble

The method of Noble ((VeW) 6.2.4), (ArA) 4.2), is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function W(·,·) which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept)

Let $(E,\langle,\rangle)$ and $(E',(,))$ be Hilbert spaces and $T: E \rightarrow E'$, $T^*: E' \rightarrow E$ linear operators fulfilling $(u', Tu) = \langle T^*u', u\rangle$ and let $W: E'xE \rightarrow R$ a functional fulfilling

$$T = \frac{\partial W(u',)}{\partial u'} \quad \text{and} \quad T^* = \frac{\partial W(.,u)}{\partial u}$$

i.e., the operators T and T^* are deviations from $W(.,.)$ in the sense of Gateaux, i.e.

$$lim \frac{F(u+tv)-F(v)}{t} = F_u(v) \text{ for all } v \in E \ .$$

Putting $W(u',u) := \frac{1}{2}(u',u') - F(u)$ the minimization problem

$$(*) \qquad J(u) := (Tu, Tu) + 2F(u) \rightarrow min \quad , \quad u \in U \subset E$$

leads to $Tu = u'$ and $(T^*u',.) = -F_u(.)$ and therefore to

Lemma A.2 (method of Noble): If $F(.)$ is a convex functional it follows that $W(u',u)$ is convex concerning u' and concave concerning u. The minimization problem (*) is equivalent to the variational equation

$$(v', T\phi) + F_u(\phi) = 0 \text{ for all } \phi \in U \quad \text{resp.} \quad (T^*v', \phi) = -F_u(\phi) \text{ for all } \phi \in U.$$

i.e., there is a characterization of the solution of (*) as a saddle point.

The Hilbert spaces H_α, $H_{(\tau)}$, $H_\alpha \otimes H_{\alpha.(\tau)}$

For the technical details we refer to the appendix B. Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. Then Hilbert spaces $\{H_\alpha | \alpha \in R\}$ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \, , \, x_n := (x, \varphi_n).$$

In case of $\alpha = 0$ we skip the subscript.The bilinear form $a(x,y) := (Ax, y)$ defines an inner (kinetic energy) product in $D(A) = H_1$ and the operator equation $Ax = f$ is equivalent to, (BrK),

$$(x,y)_1 = (f,y), \forall y \in H_1.$$

For $\alpha < 0$ the Fourier coefficients x_n contribute to the α -norm with a polynomial decay. For $\tau > 0$ the inner product resp. norm in the form

$$(x,y)_{(\tau)} = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n y_n \, , \, \|x\|_{(\tau)}^2 = (x,x)_{(\tau)}$$

spanning the Hilbert space $H_{(\tau)}$ have an exponential decay with

$$\|x\|_{(\tau)}^2 \leq c(\alpha, \tau) \|x\|_\alpha^2, \forall x \in H_\alpha.$$

The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(\tau)}^2 \text{ with } \alpha, \delta > 0 \text{ being arbitrary.}$$

Especially for $\alpha = 1/2$ one get

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{\tau/\delta} \|x\|_{(\tau)}^2 \text{ with } \delta > 0 \text{ being arbitrary.}$$

Putting

$$\|x\|_{\alpha.(\tau)}^2 := \sum_{n=1}^\infty \lambda_n^\alpha e^{-\sqrt{\lambda_n}\tau} x_n^2$$

one gets, (appendix, (BrK1))

i) $\int_0^\infty \|x\|_{(\tau)}^2 d\tau = \sum_{n=1}^\infty \lambda_n^{-1/2} x_n^2 = \|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{\tau/\delta} \|x\|_{(\tau)}^2$ for $\delta > 0$

ii) $(\ddot{x}, x)_{(\tau)} = \|\dot{x}\|_{(\tau)}^2 = \frac{1}{4} \sum_{n=1}^\infty \lambda_n e^{-\sqrt{\lambda_n}\tau} x_n^2 = \frac{1}{4} \|x\|_{1/2}^2$

iii) $\int_0^\infty \|\dot{x}\|_{(\tau)}^2 d\tau = \frac{1}{4} \sum_{n=1}^\infty x_n^2 = \frac{1}{4} \|x\|_0^2$.

Remark: We note that the D'Alembert operator with domain $L_2(H_{\alpha.(\tau)})$ is a strongly hyperbolic operator.

Let $\Phi_n := \varphi_n^H$ denote the Hilbert transform of φ_n with $(\varphi_n, \Phi_n) = 0$, (BrK1). The Hilbert space H_α of the composition $H_\alpha \otimes H_{\alpha.(\tau)}$ is built by the orthogonal system $\{\varphi_n\}$ while the Hilbert space $H_{(\tau)}$ is built by the orthogonal system $\{\Phi_n\}$ equipped with the related inner products resp. norms in the form

$$(x,y)_\alpha = \sum_{n=1}^\infty \lambda_n^\alpha x_n^{kin} y_n^{kin}, \qquad \|x\|_\alpha^2 = (x,x)_\alpha, \qquad x_n^{kin} := (x, \varphi_n), \;\; \alpha \in R$$

$$(x,y)_{\alpha.(\tau)} = \sum_{n=1}^\infty \lambda_n^\alpha e^{-\sqrt{\lambda_n}\tau} x_n^{pot} y_n^{pot}, \quad \|x\|_{\alpha.(\tau)}^2 = (x,x)_{\alpha.(\tau)}, \qquad x_n^{pot} := (x, \Phi_n), \;\; \tau > 0.$$

In the following we shall omit the Fourier coefficient indices refering to the related *kin*etic and *pot*ential energy norm case.

Then, the system $\left\{\psi_{n.\alpha.\tau}^{(1)}, \psi_{n.\alpha.\tau}^{(2)}\right\}$ with

$$\psi_{n.\alpha.\tau}^{(1)} := \lambda_n^{\alpha/2}\varphi_n - i\lambda_n^{\alpha/2}\Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}, \quad \psi_{n.\alpha.\tau}^{(2)} := \lambda_n^{\alpha/2}\varphi_n + i\lambda_n^{\alpha/2}\Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}$$

defines an orthogonal system of the Hilbert space composition $H_\alpha \otimes H_{\alpha.(\tau)}$. For

$$x_{\alpha.\tau}^{(1)} := \sum_{n=1}^\infty x_n \psi_{n.\alpha.\tau}^{(1)}, \quad x_\tau^{(2)} := \sum_{n=1}^\infty x_n \psi_{n.\alpha.\tau}^{(2)}$$

the corresponding inner product of $H_\alpha \otimes H_{\alpha.(\tau)}$ is given by

$$\left(x_{\alpha.\tau}^{(1)}, x_{\alpha.\tau}^{(2)}\right) = (x,y)_\alpha + (x,y)_{\alpha.(\tau)}.$$

The relationship between the norms above and there relationship to the statistical L_2 norm is given by

$$\int_0^\infty \|x\|_{\alpha.(\tau)}^2 d\tau = \sum_{n=1}^\infty \lambda_n^\alpha \lambda_n^{-1/2} x_n^2 = \|x\|_{\alpha-1/2}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2 \text{ for } \delta > 0$$

which is a consequence from (appendix)

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2$$

with $\delta > 0$ being arbitrary.

The Krein space $H_{(\tau)} = H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$

The Hilbert space $H_{(\tau)}$ decomposition in the form

$$H_{(\tau)} = H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$$

is supposed to be a quanta potential Hilbert-Krein space framework, where the parameter κ relates to correspondingly defined quantum number sequences in the form

$$\kappa^{+}_{\tau.n} := \frac{1}{2}\frac{e^{\kappa_n \tau}}{\cosh(\kappa_n \tau)} , \kappa^{-}_{n} := \frac{1}{2}\frac{e^{-\kappa_n \tau}}{\cosh(\kappa_n \tau)} \text{ with } \kappa_n \in R.$$

For

$$x_{(\tau)} := \sum_{n=1}^{\infty} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H_{(\tau)}$$

$$x^{+}_{\kappa.(\tau)} := \sum_{n=1}^{\infty} \kappa^{+}_{\tau.n} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H^{+}_{\kappa.(\tau)}$$

$$x^{-}_{\kappa.(\tau)} := \sum_{n=1}^{\infty} \kappa^{-}_{\tau.n} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H^{-}_{\kappa.(\tau)}$$

it follows (*)

$$x_{(\tau)} = x^{+}_{\kappa.(\tau)} + x^{-}_{\kappa.(\tau)} .$$

The Hilbert space decomposition $H_{(\tau)} = H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$ is accompanied by the indefinite inner products resp. metric

$$[x,y]_{\kappa.(\tau)} := \left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) - \left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{sinh\,(2\kappa_n \tau)}{cosh^2(\kappa_n \tau)} e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

$$= \sum_{n=1}^{\infty} tanh\,(\kappa_n \tau) e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

where

$$\left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) := \sum_{n=1}^{\infty} (\kappa^{+}_{\tau.n})^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

$$\left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right) := \sum_{n=1}^{\infty} (\kappa^{-}_{\tau.n})^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n .$$

We note the corresponding relations in the form

$$x^{+}_{\kappa.(\tau)} - x^{-}_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh(\kappa_n \tau) e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n$$

$$\left\|x^{+}_{\kappa.(\tau)}\right\|^2 - \left\|x^{-}_{\kappa(\tau)}\right\|^2 = \sum_{n=1}^{\infty} \tanh(\kappa_n \tau)\, e^{-\sqrt{\lambda_n}\tau} x_n^2 = [x,x]_{\kappa.(\tau)} .$$

From the equivalent formulas

$$(x,y)_{(\tau)} = \left[x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right] - \left[x^{-}_{\kappa.(\tau)}, y^{-}_{\kappa.(\tau)}\right]$$

$$[x,y]_{\kappa.(\tau)} := \left(x^{+}_{\kappa.(\tau)}, y^{+}_{\kappa.(\tau)}\right) - \left(x^{-}_{\kappa(\tau)}, y^{-}_{\kappa.(\tau)}\right)$$

it follows the characterization of „positive“, „negative“, and „neutral“ vectors $x \in H_{(\tau)}$ by the relations

$$\left\|x^{+}_{\kappa.(\tau)}\right\| > \left\|x^{-}_{\kappa(\tau)}\right\| \;,\; \left\|x^{+}_{\kappa.(\tau)}\right\| < \left\|x^{-}_{\kappa.(\tau)}\right\| \;,\; \left\|x^{+}_{\kappa.(\tau)}\right\| > \left\|x^{-}_{\kappa(\tau)}\right\| .$$

(*) appendix: $\kappa^{+}_{\tau.n} + \kappa^{-}_{\tau.n} = 1$, $\kappa^{+}_{\tau.n} - \kappa^{-}_{\tau.n} = \tanh(\kappa_n \tau)$, $(\kappa^{+}_{\tau.n})^2 - (\kappa^{-}_{\tau.n})^2 = \frac{sinh\,(2\kappa_n \tau)}{cosh^2(\kappa_n \tau)} = \tanh(\kappa_n \tau)$.

The potential operator of a Krein space

The canonical J-symmetric operator of a Krein space may be intepreted as a *„potential"* operator W (VaM) p. 90. In our case it is defined by

$$W_{\kappa.\tau}x := \frac{1}{2}grad\varphi_{\kappa.\tau}(x) := x^+_{\kappa.(\tau)} - x^-_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh(\kappa_n\tau)e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n \ .$$

It is complete, invertible, isometric ($W_{\kappa.\tau} = W^{-1}_{\kappa.\tau}$) and symmetric. Thus, the bilinear form

$$((x,y))_{\kappa.(\tau)} := [W_{\kappa.\tau}x, y]_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau}x_n y_n$$

defines an inner product on all of the Hilbert space $H_{(\tau)}$ with related norm

$$|||x|||^2_{\kappa.(\tau)} := [W_{\kappa.\tau}x, x]_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau}x_n^2 \ .$$

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection" of the space $H_{(\tau)}$ in the subspace $H^+_{\kappa.(\tau)}$. The sub-space $H^+_{\kappa.(\tau)}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = 1$. The sub-space $H^-_{\kappa.(\tau)}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = -1$. The whole spectrum of $W_{\kappa.\tau}$ lies on the join of the points $\lambda = \pm 1$.

We note that the operator norm of the potential operator with respect to the inner product $(x,y)_{(\tau)}$ is equivalent to the $|||x|||^2_{\kappa.(\tau)}$, i.e., it holds

$$(W_{\kappa.\tau}x, W_{\kappa.\tau}y)_{(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau}x_n y_n = ((x,y))_{\kappa.(\tau)} \ .$$

The potential and hyperboloids of a Krein space

The indefinite metric (functional) of the considered Krein space

$$\varphi_{\kappa.\tau}(x) := [x,x]_{\kappa.(\tau)} = \left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2 = \sum_{n=1}^{\infty} tanh\,(\kappa_n\tau)e^{-\sqrt{\lambda_n}\tau}x_n^2 \ .$$

in combination with the functional $\big((x)\big) := \sqrt{\varphi_{\kappa.\tau}(x)}$ generates hyperboloids H_c, hyperbolic regions V_c, and conical region V_0 in the form

$$H_c := \{x \in H_{(\tau)} | \varphi_{\kappa.\tau}(x) = c > 0\},\ V_c := \{x \in H_{(\tau)} | \big((x)\big) \geq c > 0\},\ V_0 := \{x \in H_{(\tau)} | \big((x)\big) \geq 0\}.$$

Evidently V_c is a subspace of V_0.

(VaM) p. 91: *„If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0,\infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\left\|x^+_{\kappa.(\tau)}\right\| = \left\|x^-_{\kappa(\tau)}\right\|$ belong to K, and those points of S^1 for which $\left\|x^+_{\kappa.(\tau)}\right\| > \left\|x^-_{\kappa(\tau)}\right\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by*

$$t = c(\left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ if $\left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$."

The angular and dissipative operators of a Krein space

The counterparts of W-norms $\||x|\|_{\kappa.(\tau)}^2 := [W_{\kappa.\tau} x, x]_{\kappa.(\tau)}$ with respect to the H_α Hilbert spaces norms $\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty$ are given by the norms

$$\||x|\|_{\alpha.\kappa}^2 := \sum_{n=1}^\infty tanh^2(\kappa_n \tau)\, \lambda_n^\alpha x_n^2 .$$

Let $L := H_{\alpha.\kappa} \subset H_{(\tau)} = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$ and $P^\pm$ be the canonical projectors. Then the bounded linear operator

$$K^+ = K_{\kappa.(\tau)}^+ := P^-(P^+|H_{\alpha.\kappa})^{-1} : P^+|H_{\alpha.\kappa} \to H_{\kappa.(\tau)}^-$$

is called the angular operator for $H_{\alpha.\kappa}$ with respect to $H_{\kappa.(\tau)}^+$. Then, the set of vectors of the sub-space

$$L := H_{\alpha.\kappa} := \{x_{\alpha.\kappa}^+ + K x_{\alpha.\kappa}^+\}_{x^+ \in H_{\kappa.\alpha}^+}$$

gives the general form of all $H_{\kappa.\alpha}^+ \subset H_{\kappa.(\tau)}^+$ of the Krein space $H = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$.

Theorem 11.7 ((BoJ) p. 54, (PhR)): A subspace $L \subset H_{(\tau)} = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$ is positive if and only if the angular operator K^+ of L with respect to $H_{\kappa.(\tau)}^+$ exists and satisfies the condition

$$\left\||K_{\kappa.(\tau)}^+ x_{\kappa.(\tau)}^+|\right\|_{\kappa.(\tau)}^2 \le \left\||x_{\kappa.(\tau)}^+|\right\|_{\kappa.(\tau)}^2, \; x_{\kappa.(\tau)}^+ \in D(K_{\kappa.(\tau)}^+).$$

In particular, positive definite subspaces are characterized by the property

$$\left\||K_{\kappa.(\tau)}^+ x_{\kappa.(\tau)}^+|\right\|_{\kappa.(\tau)}^2 < \left\||x_{\kappa.(\tau)}^+|\right\|_{\kappa.(\tau)}^2, \; x_{\kappa.(\tau)}^+ \in D(K_{\kappa.(\tau)}^+), \; x_{\kappa.(\tau)}^+ \ne 0,$$

and neutral subspaces by

$$\left\||K_{\kappa.(\tau)}^+ x_{\kappa.(\tau)}^+|\right\|_{\kappa.(\tau)}^2 = \left\||x_{\kappa.(\tau)}^+|\right\|_{\kappa.(\tau)}^2, \; x_{\kappa.(\tau)}^+ \in D(K_{\kappa.(\tau)}^+).$$

The inclusion $H_{\kappa.\alpha}^+ \subset H_{\kappa.(\tau)}^+$ is accompanied by related inclusions $H_{\kappa.\alpha}^- \subset H_{\kappa.(\tau)}^-$. The related Krein space concept is called alternating (maximal) pairs and alternating extensions.

The physical application of maximal positive and negative sub-spaces is concerned with the concept of of maximal dissipative (and maximal accretive) operators accompanied with spectra of unitary and self-adjoint operators, (BoJ) p. 114 ff.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $D(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1 x_1, x_1] + [x_1, T_1 x_1] \le 0, \; x_1 \in D(T_1)$$

$$[T_2 x_2, x_2] + [x_2, T_2 x_2] \le 0, \; x_2 \in D(T_2).$$

Mean ergotic theorem: If U is an isometry on a complex Hilbert space and if P is a projection on the space of all vectors invariant under U, then $\frac{1}{n}\sum_{j=0}^{n-1} U^j x$ converges to Px for every x in the space.

Krein spaces, potentials and potential operators
(AzT), (BoJ)
For further cited references we refer to (BrK1)

A Krein space is a Hilbert space H with inner product (x, y), which can be written in the form $H = H^+ \otimes H^-$. There are two equivalent approaches defining Krein spaces based on

the concept of an indefinite metric (also called a Q-metric) $Q(x, y) \coloneqq [x, y]$, $\forall x, y \in H$

a self-adjoint operator B defined on all of the Hilbert space H inducing the decomposition of H.

A canonical decomposition of $H = H^+ + H^-$ enables the (positive definite) inner product of H according to

$$(*) \;\; (x, y) = [x^+, y^+] - [x^-, y^-], x = x^+ + x^-, y = y^+ + y^-.$$

For vectors $u, v \in H^+$ we have $(u, v) = [u, v]$; for vectors $u, v \in H^-$ we have $(u, v) = -[u, v]$. If $u \in H^+$ and $v \in H^-$, then it follows from (*) that $(u, v) = [u,, \theta] - [, \theta, v]$.

The formula (*) can be inverted in the following way

$$[x, y] = (x^+, y^+) - (x^-, y^-) \text{ resp. } [x, x] = (x^+, x^+) - (x^-, x^-)$$

from which it follows

„Positivity, negativity, and neutrality of a vector $x \in H$ are equivalent to the relations

$\|x^+\| > \|x^-\|$, $\|x^+\| < \|x^-\|$, or $\|x^+\| > \|x^-\|$ respectively."

In short, a Krein space can be looked on as an arbitrary Hilbert space decomposed into usual orthogonal sums of two subspaces, equipped in addition to the original Hilbert metric (i.e., the inner product (x, y)) with an additional indefinite metric $[x, y]$.

The decomposition of a Krein space generates two mutually complementary projectors P^+ and P^- mapping H on to H^+ and H^- respectively. Those orthogonal projection operators P^+ and P^- are linked to the indefinite metric by, (VaM) chapter IV,

$$\varphi(x) := [x, x] = \|P^+ x\|^2 - \|P^- x\|^2 .$$

The indefinite metric $\varphi(x)$ can be interpreted as a „potential". The related „potential operator" (in mathematics it is called „the canonical symmetry" J, (AzT) §3, (BoJ) p. 52) is then given by, (VaM) (10.7), (12.6)

$$\boldsymbol{W}(x) := \frac{1}{2} grad\varphi(x) = P^+ x - P^- x = x^+ - x^-.$$

The fundamental properties of the potential operator $\boldsymbol{W}(x)$ are completeness, invertibility, ($\boldsymbol{W} = \boldsymbol{W}^{-1}$) isometry, and symmetry. Thus, the bilinear form $(x, y)_W \coloneqq (\boldsymbol{W}(x), y)$ defines an inner product, (BoJ) p. 52.

The sub-space H^+ is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = 1$.

The sub-space H^- is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = -1$.

The whole spectrum of $\boldsymbol{W}$ lies on the join of the points $\lambda = \pm 1$.

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection" of the space H in the subspace H^+ .

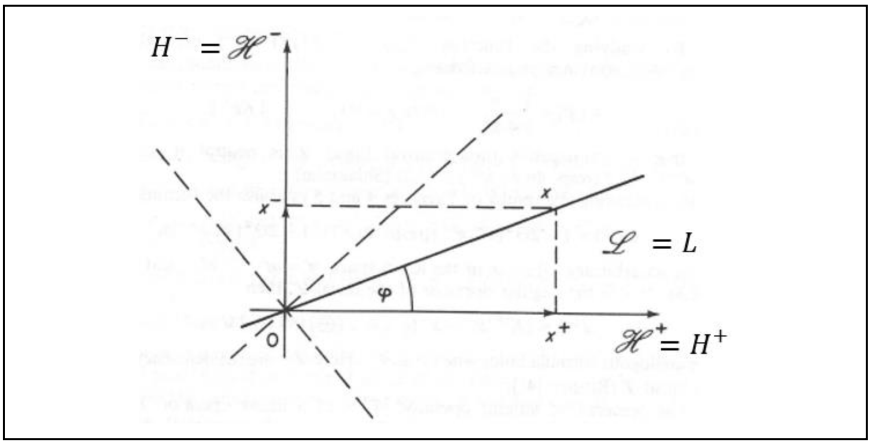
Krein spaces and angular (dissipative and accretive) operators
(AzT), (BoJ)

By the aid of J-norms a description of semi-definite subspaces L can be given enabling the definition of an angular operator $K^+ : H^+ \to H^-$ with domain $D(K^+) = P^+(L)$ and range $R(K^+) = P^-(L)$, (BoJ) p. 54. For the following we refer to (AzT) p. 48 ff. and (BoJ) p. 54.

Let $L \subset H$ in a Krein space $H = H^+ \otimes H^-$ and $P^\pm$ the canonical projectors (. Then the bounded linear operator

$$K^+ := P^-(P^+|L)^{-1} : P^+|L \to H^-$$

is called the angular operator for L with respect to H^+. The meaning of this nomenclature is explained by the following picture, (AzT) p. 61:



In the figure above a non-negative (even positive) subspace $L \subset H$ is shown. For any $x \in L$ we have $x = x^+ - x^-$, and $x^- = Kx^+$, where K is the operator of rotating the vector x^+ through an angle $\pi/2$ (in the positive direction), and then multiplying by a scalar $k = tan\varphi$ - the angular coefficient of the „line" L . …
If φ is always understood to be the *minimal* angle between L and „the axis" H^+, then $\tan(\varphi) = \|K\|$. In the general case too ($dimH \leq \infty$) for the angular operator K of a non-negative subspace L we have $tan\ (\varphi(L, H^+) = \|K\|$, if the (minimal) angle φ is defined by the equality $sin\ (\varphi(L, H^+) = sup_{e \in S(L)} \|e - ZP^+e\|$, where $S(L)$ is the unit sphere of the lineal L ($\|e\| = 1$).

Theorem 8.2 ((AzT) p. 49; see also Theorem 11.6, (BoJ) p. 54): The set of vectors

$$L = \{x^+ + Kx^+\}_{x^+ \in L^+}$$

in which L^+ is an arbitrary lineal from H^+, and $K : L^+ \to H^-$ is an arbitrary compression ($\|K\| \leq 1$), gives the general form of all $L \subset H$ of the Krein space $H = H^+ \otimes H^-$, and $L^+ = P^+(L)$ and K is the angular operator for L with respect to H^+.

Let $\|x\|_W^2 = \|x\|_J^2 = \|x^+\|^2 - \|x^-\|^2$ denote the $J = W$-inner product related (potential) norm.

Theorem 11.7 ((BoJ) p. 54): A subspace $L \subset H$ is positive if and only if the angular operator K^+ of L with respect to H^+ exists and satisfies the condition

$$\|K^+x^+\|_W^2 \leq \|x^+\|_W^2 , x^+ \in D(K^+).$$

In particular, positive definite subspaces are characterized by the property

$$\|K^+x^+\|_W^2 < \|x^+\|_W^2 , x^+ \in D(K^+), x^+ \neq 0,$$

and neutral subspaces by

$$\|K^+x^+\|_W^2 = \|x^+\|_W^2 , x^+ \in D(K^+).$$

For negative subspaces similar statements, involving K^- instead of K^+, are valid.

Theorem 8.2' ((AzT) p. 49): The set of vectors

$$L = \{Qx^- + x^-\}_{x^- \in L^-}$$

in which L^- is an arbitrary lineal from H^-, and $Q : L^- \rightarrow H^+$ is an arbitrary compression ($\|Q\| < 1$), gives the general form of all $L^+ \subset H$ of the Krein space $H = H^+ \otimes H^-$, and $L^- = P^-(L)$ and Q is the angular operator for L with respect to H^-.

Alternating pairs and dissipative operators in Hilbert space

(BoJ)

(BoJ) p. 39: Let H_0 denote a Hilbert space with with inner product $(x, y)_0$, $x, y \in H_0$ and norm $\|x\|$ and let W be an arbitrary bounded self-adjoint operator ($W = W^*$) given on H_0. Then the Hermitian sesquilinear form $[x, y] = (Wx, y)_0 = Q(x, y)$ defines in H_0 an indefinite metric which we shall call the W-metric, and we shall call the space H_0 itself with the W-metric a W-space. W is called the Gram operator of the space H_0.

(BoJ) p. 91: A linear operator A with an arbitrary domain of definition $D(A)$, operating in a W-space H_0, is said to be W-dissipative if $Im[Ax, x] \geq 0$ for all $x \in D(A)$, and to be maximal W-dissipative if it is W-dissipative and coincides with any W -dissipative extension of it.

An ordered pair of subspaces $\{L_1, L_2\}$ of the Krein space H will be called an alternating pair provided L_1 is positive, L_2 is negative, and $L_1 \perp L_2$. If, in addition, L_1 is maximal positive and L_2 is maximal negative, the pair $\{L_1, L_2\}$ is called alternating maximal pair.

By an alternating extension of the alternating pair $\{L_1, L_2\}$ we mean an alternating pair $\{L'_1 L'_2\}$ such that $L_1 \subset L'_1$, $L_2 \subset L'_2$.

Theorem 9.1 (BoJ) p. 115: Every alternating pair in the Krein space H can be extended to an alternating maximal pair.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $D(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1x_1, x_1] + [x_1, T_1x_1] \leq 0, x_1 \in D(T_1)$$

$$[T_2x_2, x_2] + [x_2, T_2x_2] \leq 0, x_2 \in D(T_2).$$

Theorem (BoJ) p. 118: If $\left\{L_1^{(0)}, L_1^{(0)}\right\}$ is an alternating maximal pair extending $\{D(-T_1), D(-T_2)\}$, then the operators $T_1^{(0)}, T_2^{(0)}$ defined by the relations $L_1^{(0)} = D(-T_1^{(0)})$, $L_2^{(0)} = D(-T_2^{(0)})$ are maximal dissipative operators of the dissipative operators T_1, T_2, and every solution can be obtained in this way.

Krein spaces and hyperboloids
accompanied by hyperbolic and conical regions
(VaM) p. 89 ff.

Putting $x^+ := P^+x$, $x^- := P^-x$ the corresponding potential $\varphi(x^+ + x^-)$ defined by

$$\varphi(x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0$$

generates hyperboloids in the form

$$H_c := \{x \in H | (x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0\}.$$

A hyperbolic region V_c is defined by

$$\big((x)\big) = \sqrt{\|x^+\|^2 - \|x^-\|^2} \geq c > 0$$

A conical region V_0 is defined by

$$\big((x)\big) = \sqrt{\|x^+\|^2 - \|x^-\|^2} \geq 0 \ .$$

Evidently V_c is a subspace of V_0.

If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0, \infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0, \infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 \ = 1$), then those points of S^1 for which $\|P^+x\| = \|P^-x\|$ belong to K, and those points of S^1 for which $\|P^+x\| > \|P^-x\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by

$$t = \mathrm{c}(\|x^+\|^2 - \|x^-\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ if $\|x^+\|^2 - \|x^-\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$.

Lemma: If the (proper) subspace $H_1 \subset H$ is finite dimensional, then the region V_c ($c \geq 0$) is weakly closed.

Remark: Ellipsoids are defined by the condition $\frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} = 1$. The related elliptical region is defined by

$$E_c := \left\{x \in H | \frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} \leq c, c > 0\right\}.$$

Theorem (ZaC) p. 291: Let H denote a Hilbert space with inner product $(\cdot,\cdot)$ and $K \subset H$ be a closed convex cone. For every $x \in H$ let $P^K x$ (which is uniqly defined) denote the projection of x on K. Putting $K^- := -K^+ := \{y \in H] \ (x, y) \leq 0, \forall x \in H\}$ it holds $\forall x \in H \ x = P^K x + P^{K^-} x$ and $(P^K x, P^{K^-} x) = 0$. Conversely, if $x = x_1 + x_2$ with $x_1 \in K$, $x_2 \in K^-$ and $(x_1, x_2) = 0$ then $x_1 = P^K x$ and $x_2 = P^{K^-} x$.

Hyperboloids generated by operators
(VaM) p. 92

Let B be self-adjoint operator defined on all of the Hilbert space H. Since it follows that B is bounded, then

$$inf\{\,(Bx,x) = a \mid \|x\| = 1\} > \infty\ ,\ sup\{\,(Bx,x) = b \mid \|x\| = 1\} < \infty\ .$$

We shall assume that $a < 0$, $b > 0$. Further, let E_t be the resolution of the odentity corresponding to B; then $E_b - E_0 = P_1$ is a projection operator onto subspace $H_1 \subset H$ which reduces B. Thus, the operator induces a decomposition of into the direct sum of subspaces H_1 and H_2 ($H = H_1 \otimes H_2$) and thereby generated a hyperboloid

$$\varphi(x) = \varphi(x^+ + x^-) = \sqrt{\|P_1\|^2 - \|P_2\|^2} = c > 0\ ,$$

where P_2 is the projection onto H_2.

In this case where the positive part of the spectrum of B lies in an interval $[m, b]$, where $m > 0$, then the inequality

$$\|Bx\| \geq \frac{m}{\sqrt{2}}\sqrt{\varphi^2(x) + \|x\|^2} \geq \frac{m}{\sqrt{2}}\sqrt{c^2 + \|x\|^2}$$

holds for every x in the hyperbolic region V_c defined by

$$\varphi(x) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq c > 0\ ,$$

as well as in the conical region V_0 defined by

$$\varphi(x) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq 0\ .$$

Remark: It should be remarked that in some cases the operator B leaves invariant the hyperbolic regions V_c, which it generates. This is the case, for example, when the positive part of the spectrum of B lies in the interval $[1, b]$ and the negative part lies in $[-1,0]$. In fact, we then have

$$\big((Bx)\big) = \|P^+Bx\|^2 - \|P^-Bx\|^2 = \|BP^+x\|^2 - \|BP^-x\|^2$$

$$= \int_1^b t^2 d(E_t\, P^+x, P^+x) - \int_{-1}^0 t^2 d(E_t\, P^-x, P^-x)$$

$$\geq \|P^+x\|^2 - \|P^-x\|^2 \geq c^2.$$

The telegraph equation
(CoR) p. 192 ff.

For the wave equation

$$\frac{1}{c^2}\ddot{u} - \Delta u = 0\,,$$

progressing undistorted plane waves with speed c and the arbitrary form

$$\Phi(\sum_{i=1}^{n}\alpha_i x_i - ct),\ \sum_{i=1}^{n}\alpha_i^2 = 1$$

are possible in every direction. A more general example is given by the telegraph equation

$$\ddot{u} - c^2 u'' + (\alpha+\beta)\dot{u} + \alpha\beta u = 0,$$

satisfied by the voltage or the current u as a function of the time t and the position x along a cable; here x measures the length of the cable from an initial point. Unless this equation represents dispersion. If we introduce $v := e^{\frac{1}{2}(\alpha+\beta)t}u$, we obtain the simpler equation

$$\ddot{v} - c^2 v'' + (\frac{\alpha-\beta}{2})^2 v = 0$$

for the function v. This new equation represents the dispersionless case if and only if $\alpha = \beta$. In this case the original telegraph equation, of course, possesses no absolutely undistorted wave solutions of arbitrarily prescribed form. However, our result may be stated in the following way:

> If condition $\alpha = \beta$ holds, the telegraph equation posses damped, yet „relatively“ undistorted, progressing wave solutions of the form $u = e^{-\frac{1}{2}(\alpha+\beta)t} f(x \pm ct)$, with arbitrary f, progressing in both directions of the cable.

The telegraph equation

$$\ddot{u} - c^2 u'' + (\alpha+\beta)\dot{u} + \alpha\beta u = 0,$$

is derived by elimination of one of the unknown functions from the following system of two differential equations of first order for the current $i = i(x,t)$ and the voltage $u = u(x,t)$ as functions of x and t:

$$C\dot{u} + Gu + i' = 0$$

$$L\dot{i_t} + Ri + u' = 0.$$

Here L is the inductance of the cable, R its resistence, C its shunt capacity, and, finally, G, its shunt conductance (loss of current divided by voltage). The constants in the telegraph equation, which arise in the elimination process, have the meaning

$$\frac{1}{c^2} = LC,\ \alpha = \frac{G}{C},\ \beta = \frac{R}{L}$$

where c is the speed of light and α the capacitive and β the inductive damping factor.

Global boundedness of the 3D-Navier-Stokes equations
in a variational $H_{-1/2}$ based Hilbert space framework
(BrK1)
www.navier-stokes-equations.com

It turned out that based on the physical modelling assumption of a variational representation of the 3D NSE in a $H_{-1/2}$ Hilbert space framework (interpreted as a fluid element test space) the 3D NSE enjoy global solutions. Its a consequence of the well-known Sobolevskii-estimates for the 3D case. Those estimates fail in case of a H_0 test space.

Lemma (Sobolevskii): For $0 \le \delta < 1/2 + n \cdot (1 - 1/p)/2$ it holds

$$\left|A^{-\delta} P(u, grad)v\right|_p \le M \cdot \left|A^{\theta} u\right|_p \cdot |A^{\rho} u|_p$$

with a constant $M := M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \ge n/2p + 1/2$, $\theta, \rho > 0, \theta + \rho > 1/2$.

The NSE initial-boundary equation is given by

$$\frac{du}{dt} + Au + Bu = Pf \text{ , } u(0) = u_0$$

where $B(u) := P(u, grad)u)$ and $Pu_0 = u_0$. Multiplying this homogeneous equation with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_\alpha + (Au, u)_\alpha + (Bu, u)_\alpha = 0, \ (u(0), v)_\alpha = (u_0, v)_\alpha \text{ for all } v \in H_{-1/2}.$$

For $\alpha := -1/2$ one gets the generalized "energy" inequality in the form

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

Putting $p := 2$, $\delta := 1/4$,$\theta := \rho := 1/2$ fulfilling $\theta + \rho \ge \frac{1}{4}(n+1) = 1$ one gets from the Sobolevskii-lemma above

$$\left\|A^{-\delta} P(u, grad)u\right\| \le c\left\|A^{\theta} u\right\| \cdot \|A^{\rho} u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\},$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

Remark: We note that the pressure p in the variational representation

$$(Au, v)_{-\frac{1}{2}} \coloneqq (\nabla u, \nabla v)_{-\frac{1}{2}} + (\nabla p, v)_{-\frac{1}{2}} = (u, v)_{\frac{1}{2}} + (p, v)_0 \qquad \text{for all } v \in H_{-1/2}$$

$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2}$$

can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^3 R_j R_k (u_j u_k)$$

with (R_1, R_2, R_3) is the Riesz transform.

Some formulas

(Grl)

i) $cosh(x) \pm sinh(x) = e^{\pm x}$

ii) $cosh^2(x) - sinh^2(x) = 1$

iii) $tanh(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx},\ x > 0$ (Grl) 1.232

iv) $tanh(x) = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)}{(2k)!}B_{2k}x^{2k-1}$

v) $e^{ax} - e^{bx} = (a-b)xe^{\frac{1}{2}(a+b)x}\prod_{k=1}^{\infty}(1 + \frac{(a-b)^2x^2}{4k^2\pi^2}$ (Grl) 1.223

vi) $sinh(2x) = 2\,sinh(x)\,cosh(x),\ cosh(2x) = 2cosh^2(x) - 1$ (Grl) 1.334

vii) $tanh(x)\frac{sinh\,(2x)}{cosh^2(x)} = tanh(x)\frac{2\,sinh(x)cosh(x)}{cosh^2(x)} = 2\,tanh^2(x)$

viii) $\int sinh(ax)dx = \frac{1}{a}cosh(ax)\quad ,\ \int cosh(ax)dx = \frac{1}{a}sinh(ax)$ (Grl) 2.414

ix) $\int\frac{dx}{cosh^2(x)} = tanh(x)\quad ,\int\frac{dx}{sinh^2(x)} = -\,coth(x)$ (Grl) 2.422

x) $\int tanh(x)dx = ln(cosh(x))\quad ,\int coth(x)dx = ln(sinh(x))$ (Grl) 2.423

xi) $\int\frac{\sinh\,(2nx)}{\cosh\,(x)}dx = 2\sum_{k=0}^{n-1}(-1)^k\frac{\cosh((2n-2k-1)x)}{2n-2k-1}$ (Grl) 2.433

xii) $\int\frac{sinh\,(2x)}{cosh\,(x)}dx = 2cosh(x)$

xiii) $\int tanh(x)dx = ln(cosh(x))\quad ,\int coth(x)dx = ln(sinh(x))$ (Grl) 2.423

xiv) $\int_0^{\infty} e^{-\alpha x}\,tanh(x)\,dx = \beta\left(\frac{\alpha}{2}\right) - \frac{1}{\alpha}\quad ,\ Re(\alpha) > 0$ (Grl) 3.541

xv) $a^2 \neq b^2,$ (Grl) 2.481

$$\int e^{ax}sinh\,(bx+c)dx = \frac{e^{ax}}{a^2-b^2}[a\cdot sinh\,(bx+c) - b\cdot cosh(bx+c)]$$

$$\int e^{ax}cosh\,(bx+c)dx = \frac{e^{ax}}{a^2-b^2}[a\cdot cosh\,(bx+c) - b\cdot sinh(bx+c)]$$

xvi) $a^2 \neq b^2,$ (Grl) 2.484

$$\int e^{ax}sinh\,(bx)\frac{dx}{x} = \frac{1}{2}\{[Ei(a+b)x] - [Ei(a-b)x]\}$$

$$\int e^{ax}cosh\,(bx)\frac{dx}{x} = \frac{1}{2}\{[Ei(a+b)x] + [Ei(a-b)x]\}$$

xvii) $tanh(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx}$ (Grl) 1.232

xviii) $e^{ax} - e^{bx} = (a-b)xe^{\frac{1}{2}(a+b)x}\prod_{k=1}^{\infty}(1 + \frac{(a-b)^2x^2}{4k^2\pi^2}$ (Grl) 1.223

xix) $\int_0^{\infty} e^{-zx}\tanh(x)\,dx = \beta\left(\frac{z}{2}\right) - \frac{1}{z},\ \beta(1) = log2,\ \beta\left(\frac{1}{2}\right) = \frac{\pi}{2},\ Re(z) > 0$ (Grl) 3.541

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