A Krein space based quantum potential energy model for an integrated Plasma and Mie theory

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Dedicated to my son Mario
on the occasion of his 31th birthday, Dec 2, 2022

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M. Heidegger*: "modern physics is called mathematical because, in a remarkable way, it makes use of a quite specific mathematics. But it can proceed mathematically in this way only because, in a deeper sense, it is already itself mathematical".*

F. Ehrenhaft: *„light beams must have electric stationary components in the direction of the wave front normal, and that consequently there must be stationary electric potential differences between different points along the beam ; and that there must be also a stationary magnetic field in the beam of light with potential differences. Hence, the light beam must have a magnetizing effect, and the charge of a magnet should be changed by light", (EhF1),* (*).

Abstract: A Krein space based matter field theory is provided. From the Mie theory the concept of discrete energy knots is taken modelled by a physical problem specific (self-adjoint) kinetic energy operator. From the correspondingly defined Krein space framework the concept of a (self-adjoint) potential energy operator is applied. It enables the definition of κ-potential energy norms on all of the Krein space built on sets of κ-quantum numbers leading to a corresponding (vacuum, plasma, Mie) grouping of the concerned κ-quantum elements.

Key words: integrated Plasma and Mie theory, integrated κ-quantum number systems, self-adjoint potential energy operators

(*)
(EhF) p. 238: The speed of the magneto-photophoresis depends upon the intensity and the frequency of the light, and upon the intensity of the magnetic field as well as on the material itself.
(EhF) p. 238: The Trembling-Effect: This rapidly changing movement of diamagnetic particles in low fields cannot be confused with Brownian movement in gases.
(EhF) p. 240: This geomagnetic field alone without any artificial field caused nickel particles to move up.
(EhF) p. 242: The prevailing opinion is that within an arbitrarily chosen geometric surface a real quantity of either kind of electricity can be inclosed, but no matter how the surface is chosen, it will always inclose the same amount of north and south magnetism. In other words there are true quantities of electricity of either sign, but not true magnetic quantities. Thus we have electric but not magnetic currents.
(EhF) p. 243-244: The light induces electric and magnetic charges (poles) upon the particles if they are illuminated by concentrated light pre-ponderantly shorter wave lengths. … For the magnetic charges this conclusion is new, but is justified because of the complete analogy of this phenomenon with the electric phenomenon.

1. Introduction

About 95% of the universe is about the phenomenon „vacuum“. The same proportion applies to the emptyness between a proton and an electron. The remaining 5% of universe's vacuum consists roughly of 5% matter, of 25% dark matter, and of 70% dark energy. Nearly all (about 99%) of the 5% matter in the universe is in "plasma state". A presumed physical concept of „dark matter“ „explains“ the phenomenon of the spiral shapes in the universe. A presumed physical concept of „dark energy“ explains the phenomenon of the cosmic microwave background.

The SMEP provides a collection of assumed physical elementary particles grouped according to the three physical forces phenomena, the electromagnetic force, the weak and the strong interaction forces. Their interacting is governed by required (conservation of energy) symmetry groups. The specification attributes of those three independent „standard elementary particle models“ are constructed according to the fundamental concept of Dirac's theory of radiation, (appendix B):

> *E. Fermi: „Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field“.*

Conceptually, it is the *„small term representing the coupling energy of the atom and the radiation field“*, which requires the three independent symmetry groups of the SMEP with force type specific elementary particle split into fermions and bosons jeopardizing all developments of the SMEP to an unified field theory.

In the proposed Krein space based quantum potential model the potential operator is an intrinsic part of the Krein space framwork and not a physical phenomenon specific appropriately chosen function, like the Coulomb potential function in Dirac's theory of radiation. The (complementary) kinetical energy system of this model is defined by the energy knots of a physical phenomenon specific kinetic energy operator. The energy knots may be interpreted as the mass of the corresponding quantum particle. The quantum elements of the proposed Krein space are composed by three elementary quantum elements, the electrino, the positrino, and the neutrino.

The model is in line with the target of the Mie theory to explain, *„why the field possesses a granular structure and why the knots of energy remain intact in spite of the back and forth flux of energy and momentum*“. The total energy of the system is given by a Hamiltonian (selfadjoint) operator H expressed as the sum of a kinetic and a potential energy operator in the form $H = E_{kin} + E_{pot}$. Until this point the model is the same as in quantum mechnics. For example, in electrostatics the kinetic operator is the (dissipative) Laplacian operator $-\Delta$ and the potential operator is given by the singular integral Coulomb potential (pseduo-differential) operator). The Krein space framework is accompanied by an intrinsic self-adjoint (J -symmetry) potential operator enabling corresponding orthogonal representation of the total energy norm in the form

$$E(x) = {}_{kin}\|x\|_1^2 + {}_{pot}\|x\|_{1/2}^2 \ , \ \ E_\kappa(x) = {}_{kin}\|x\|_1^2 + {}_{pot}\||x|\|_{1/2.\kappa}^2.$$

Therefore, the Krein space based model avoids the *„small term representing the coupling energy of the atom and the radiation field*“ of the Dirac model.

The paper is organized as follows.

In section 2 we provide the mathematical model of an integrated κ-quantum potential elements system accompanied by the concepts of a quantum potential, a quantum potential operator, an quantum angular operator, and alternating pairs of sets of quantum elements. The corresponding mathematical details for this section including supporting formulae are provided in appendix A. Appendix A also provides appropriately defined domains of the D-Alembert operator becoming a strong hyperbolic pseudo differential operator. Additionally, the crucial a priori estimate for the 3D NSE is provided enabling appropriate energy estimates with respect to $E(x)$ in the context of the related millennium problem of the Clay Mathematics Institute.

Section 3 gives the definition table of three integrated κ-quantum number systems. The table is grouped into the three physical quantum potential areas, the vaccum, the plasma, and the Mie theory quantum potential systems.

Refering to (BrK1) in the following sections the affected mathematical and physical realities are considered.

The affected mathematical realities in section 4 are split into three sub-sections, (a) primarily affected mathematical tools/equations, (b) affected millennium problems of the Clay Mathematics Institute, and (c) quadratic and complementary „least energy“ Riesz-Galerkin methods.

The affected physical realities in section 5 are split into the sub-groups, (a) microscopic and macroscopic Landau damping phenomena, (b) α, β, γ-decay phenomena, (c) the Lamb shift and the fine structure constant, and (d) other physical coupling constants, like the permeability constants ε_0 resp. μ_0 of a particle for an electric resp. magnetic field in a vacuum, superfluids, superconductors, and Bose-Einstein-Condensates, and (e) Ehrenhaft's forgotten photophoresis phenomenon.

2. The mathematical field model

Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. The Hilbert spaces $\{H_\alpha | \alpha \in R\}$ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \, , \, x_n := (x, \varphi_n)$$

accompanied by the inner product

$$(x, y)_\alpha = \sum_1^\infty \lambda_n^\alpha \, x_n \, y_n .$$

In case of $\alpha = 0$ we skip the subscript. The physical model problem for the operator A is the Friedrichs extension of the Laplacian operator $A := \overline{-\Delta}^{\| \ \|_1}$ with domain $D(A) = H_1$. Then, the bilinear form $a(u, v) := (Au, v)$ defines an inner (kinetic energy) product in $D(A) = H_1$ and the operator equation $-\Delta u = f$ is equivalent to least energy variational representation in the form, (BrK),

$$(u, v)_1 = (f, v), \forall v \in H_1 .$$

For $\alpha < 0$ the Fourier coefficients x_n contribute to the α -norm with a polynomial decay. The extended Hilbert space $H_{(\tau)}$ is defined by the inner product resp. norm

$$(x, y)_{(\tau)} = \sum_1^\infty e^{-\sqrt{\lambda_i}\tau} \, x_n \, y_n , \quad \|x\|_{(\tau)}^2 = (x, x)_{(\tau)} .$$

The (τ)-norm is weaker than any α-norm, i.e.

$$\|x\|_{(\tau)}^2 \leq c \|x\|_\alpha^2 \quad \text{for any } \alpha\text{-norm}$$

with $c = c(\alpha, \tau)$ depending only on α and τ.

Let $\Phi_n := \varphi_n^H$ denote the Hilbert transform of φ_n with $(\varphi_n, \Phi_n) = 0$, (*), (BrK1). Then, the system $\left\{\psi_{n.\tau}^{(1)}, \psi_{n.\tau}^{(2)}\right\}$ with

$$\psi_{n.\tau}^{(1)} := \varphi_n - i\Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} \, , \quad \psi_{n.\tau}^{(2)} := \varphi_n + i\Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}$$

defines an orthogonal system of the Hilbert space composition $H_0 \otimes H_{(\tau)}$.

(*) for space dimensions greater than one the counterpart of the Hilbert transform operator is the Riesz transform operator

In quantum mechanics the total energy of a system is given by a Hamiltonian (selfadjoint) operator H expressed as the sum of a kinetic and a potential energy operator in the form $H = E_{kin} + E_{pot}$.

The conceptual design of the proposed integrated mathematical model is based on a Hermitian operator expressed as the sum of two hermitian kinematical and potential operators. The domain of the kinematical energy operator is accompanied by the domain H_1. The domain of the potential energy operator is accompanied by the domains $H_{1/2}$ and $H_{1/2.\kappa}$ equipped with the norms

$$\|x\|_{1/2}^2 = \int_0^\infty \|x\|_{1.(\tau)}^2 d\tau = \sum_{n=1}^\infty \sqrt{\lambda_n} x_n^2$$

$$|||x|||_{1/2.\kappa}^2 = \sum_{n=1}^\infty \sqrt{\lambda_n} x_n^2 \int_0^\infty tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau} d\tau \,,\ \kappa_n \neq 0.$$

The related sequences

$$\kappa_{\tau.n}^+ := \frac{1}{2}\frac{e^{\kappa_n\tau}}{\cosh(\kappa_n\tau)} \,,\ \kappa_n^- := \frac{1}{2}\frac{e^{-\kappa_n\tau}}{\cosh(\kappa_n\tau)} \text{ with } \kappa_n \in R$$

define a Krein space decomposition of the Hilbert space $H_{(\tau)}$ in the form $H_{(\tau)} = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$

The Hilbert space $H_{(\tau)}$ and the Krein space $H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$ are accompanied by two related inner products on all of the Hilbert space $H_{(\tau)}$ in the form (*)

$$(x,y)_{(\tau)} = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

$$((x,y))_{\kappa.(\tau)} = \sum_{n=1}^\infty tanh^2(\kappa_n\tau)\, x_n y_n e^{-\sqrt{\lambda_n}\tau} .$$

The Krein space is also equipped with the indefinite inner products resp. metric (*)

$$[x,y]_{\kappa.(\tau)} := \left(x_{\kappa.(\tau)}^+, y_{\kappa.(\tau)}^+\right) - \left(x_{\kappa(\tau)}^-, y_{\kappa.(\tau)}^-\right) = \sum_{n=1}^\infty tanh\,(\kappa_n\tau) e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

with

$$\left(x_{\kappa.(\tau)}^+, y_{\kappa.(\tau)}^+\right) := \sum_{n=1}^\infty (\kappa_{\tau.n}^+)^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n \ ,\ \left(x_{\kappa(\tau)}^-, y_{\kappa.(\tau)}^-\right) := \sum_{n=1}^\infty (\kappa_{\tau.n}^-)^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n .$$

The indefinite norm $[x,x]_{\kappa.(\tau)}$ may be interpreted as a potential (functional), (VaM) p. 90,

$$\varphi_{\kappa.\tau}(x) := [x,x]_{\kappa.(\tau)} = \left\|x_{\kappa.(\tau)}^+\right\|^2 - \left\|x_{\kappa(\tau)}^-\right\|^2 = \sum_{n=1}^\infty \tanh(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau} x_n^2 .$$

The self-adjoint operator

$$W_{\kappa.\tau} x := x_{\kappa.(\tau)}^+ - x_{\kappa.(\tau)}^- = \sum_{n=1}^\infty tanh(\kappa_n\tau) e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n$$

may be interpreted as the quantum potential operator of the considered κ -quantum potential energy systems.

The definition of the potential operator enables a treatment of the results of its action as the „mirror reflection" of the space $H_{(\tau)}$ in the subspace $H_{\kappa.(\tau)}^+$. The sub-space $H_{\kappa.(\tau)}^+$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = 1$. The sub-space $H_{\kappa.(\tau)}^-$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = -1$. The whole spectrum of $W_{\kappa.\tau}$ lies on the join of the points $\lambda = \pm 1$.

(*) The relation to the proposed potential energy norms is given by the equality $\|x\|_{1/2}^2 = \int_0^\infty \|x\|_{1.(\tau)}^2 d\tau = \sum_{n=1}^\infty \sqrt{\lambda_n} x_n^2$.

Putting $x_{(\tau)} := \sum_{n=1}^\infty e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H_{(\tau)}$, $x_{\kappa.(\tau)}^+ := \sum_{n=1}^\infty \kappa_{\tau.n}^+ e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H_{\kappa.(\tau)}^+$, $x_{\kappa.(\tau)}^- := \sum_{n=1}^\infty \kappa_{\tau.n}^- e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H_{\kappa.(\tau)}^-$

it follows $\kappa_{\tau.n}^+ + \kappa_{\tau.n}^- = 1$, $\kappa_{\tau.n}^+ - \kappa_{\tau.n}^- = \tanh(\kappa_n\tau)$, $(\kappa_{\tau.n}^+)^2 - (\kappa_{\tau.n}^-)^2 = \frac{sinh\,(2\kappa_n\tau)}{cosh^2(\kappa_n\tau)} = \tanh(\kappa_n\tau)$

$$[x,y]_{\kappa.(\tau)} := \left(x_{\kappa.(\tau)}^+, y_{\kappa.(\tau)}^+\right) - \left(x_{\kappa(\tau)}^-, y_{\kappa.(\tau)}^-\right) = \frac{1}{2}\sum_{n=1}^\infty \frac{sinh\,(2\kappa_n\tau)}{cosh^2(\kappa_n\tau)} e^{-\sqrt{\lambda_n}\tau} x_n y_n = \sum_{n=1}^\infty tanh\,(\kappa_n\tau) e^{-\sqrt{\lambda_n}\tau} x_n y_n .$$

From the equivalent formulas

$$(x,y)_{(\tau)} = [x^+_{\kappa.(\tau)}, y^+_{\kappa.(\tau)}] - [x^-_{\kappa.(\tau)}, y^-_{\kappa.(\tau)}]$$

$$[x,y]_{\kappa.(\tau)} := (x^+_{\kappa.(\tau)}, y^+_{\kappa.(\tau)}) - (x^-_{\kappa(\tau)}, y^-_{\kappa.(\tau)})$$

it follows the characterization of „positive", „negative", and „neutral" vectors $x \in H_{(\tau)}$ by the relations

$$\|x^+_{\kappa.(\tau)}\| > \|x^-_{\kappa(\tau)}\| \ , \ \|x^+_{\kappa.(\tau)}\| < \|x^-_{\kappa.(\tau)}\| \ , \ \|x^+_{\kappa.(\tau)}\| > \|x^-_{\kappa(\tau)}\| \ .$$

The potential $\varphi_{\kappa.\tau}(x)$ in combination with the functional $((x)) := \sqrt{\varphi_{\kappa.\tau}(x)}$ generates hyperboloids H_c, hyperbolic regions V_c, and conical regions V_0 in the form

$$H_c := \{x \in H_{(\tau)} | \varphi_{\kappa.\tau}(x) = c > 0\}, V_c := \{x \in H_{(\tau)} | ((x)) \geq c > 0\}, V_0 := \{x \in H_{(\tau)} | ((x)) \geq 0\}.$$

Evidently V_c is a subspace of V_0. The boundary K of the conical region is defined by the condition $((x)) = 0$. It is an asymptotic conical manifold for the hyperboloid $((x)) = c > 0$ (*).

The counterparts of the W-norms $|||x|||^2_{\kappa.(\tau)} := [W_{\kappa.\tau}x, x]_{\kappa.(\tau)}$ with respect to the H_α Hilbert space norms $\|x\|^2_\alpha = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty$ are given by the norms

$$|||x|||^2_{\alpha.\kappa} := \sum_{n=1}^\infty tanh^2(\kappa_n \tau) \lambda_n^\alpha x_n^2 \ .$$

Let $L := H_{\alpha.\kappa} \subset H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$ and $P^\pm$ be the canonical projectors. Then the set of vectors of L can be represented in the form

$$L := H_{\alpha.\kappa} := \{x^+_{\alpha.\kappa} + K^+ x^+_{\alpha.\kappa}\}_{x^+ \in H^+_{\kappa.\alpha}}$$

giving the general form of all $H^+_{\kappa.\alpha} \subset H^+_{\kappa.(\tau)}$ of the Krein space $H = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$. The bounded linear operator

$$K^+ = K^+_{\kappa.(\tau)} := P^-(P^+|H_{\alpha.\kappa})^{-1} : P^+|H_{\alpha.\kappa} \to H^-_{\kappa.(\tau)}$$

is called the angular operator for $H_{\alpha.\kappa}$ with respect to $H^+_{\kappa.(\tau)}$ (**). The inclusion $H^+_{\kappa.\alpha} \subset H^+_{\kappa.(\tau)}$ is accompanied by related inclusions $H^-_{\kappa.\alpha} \subset H^-_{\kappa.(\tau)}$. The related Krein space concept is called alternating (maximal) pairs and alternating extensions. This concept can be applied in the context of dissipative operators in Hilbert spaces, (BoJ) p. 116.

(*) (VaM) p. 91: *„If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0,\infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $((x)) = c > 0$ in a single point.*

We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $((x)) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\|x^+_{\kappa.(\tau)}\| = \|x^-_{\kappa(\tau)}\|$ belong to K, and those points of S^1 for which $\|x^+_{\kappa.(\tau)}\| > \|x^-_{\kappa(\tau)}\|$ intersect the hyperboloid $((x)) = c > 0$ at the point whose distance from θ is given by $t = c(\|x^+_{\kappa.(\tau)}\|^2 - \|x^-_{\kappa(\tau)}\|^2)^{-1/2}$.

From this it is seen that $t \to \infty$ if $\|x^+_{\kappa.(\tau)}\|^2 - \|x^-_{\kappa(\tau)}\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $((x)) = c > 0$."

(**) The subspace $L \subset H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$ is positive if and only if the angular operator K^+ of L with respect to $H^+_{\kappa.(\tau)}$ exists and satisfies the condition

$$|||K^+_{\kappa.(\tau)} x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} \leq |||x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} \ , \ x^+_{\kappa.(\tau)} \in D(K^+_{\kappa.(\tau)}).$$

In particular, positive definite subspaces are characterized by the property

$$|||K^+_{\kappa.(\tau)} x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} < |||x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)}, \ x^+_{\kappa.(\tau)} \in D(K^+_{\kappa.(\tau)}), \ x^+_{\kappa.(\tau)} \neq 0,$$

and neutral subspaces by

$$|||K^+_{\kappa.(\tau)} x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)} = |||x^+_{\kappa.(\tau)}|||^2_{\kappa.(\tau)}, \ x^+_{\kappa.(\tau)} \in D(K^+_{\kappa.(\tau)}).$$

(***) The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $D(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1 x_1, x_1] + [x_1, T_1 x_1] \leq 0, \ x_1 \in D(T_1)$$

$$[T_2 x_2, x_2] + [x_2, T_2 x_2] \leq 0, \ x_2 \in D(T_2).$$

3. The integrated vaccum, plasma, and Mie κ-quantum potential systems

The Krein space based vaccum, plasma, and Mie κ-quantum potential systems are defined by the connected sets κ_n of related appropriatly defined sets of quantum numbers according to the following table:

Model case	EP	Anti-EP	QN quantum numbers	QN quantum numbers	QN quantum numbers
	$q^+ \in H^+_{\kappa.(\tau)}$	$q^- \in H^-_{\kappa.(\tau)}$	q_n^+	q_n^-	$\boldsymbol{\kappa_n} := q_n^+ - q_n^-$
Vacuum particles Electrino ϵ	ϵ	$\epsilon \otimes \pi \otimes \pi$	$n_\epsilon := \frac{n-1/2}{4n-1}$	$\frac{3n-1/2}{4n-1}$	$\kappa_\epsilon = -\frac{2n}{4n-1}$
Vacuum particles Positrino π	π	$\pi \otimes \epsilon \otimes \epsilon$	$n_\pi := \frac{n}{4n-1}$	$\frac{3n-1}{4n-1}$	$\kappa_\pi = -\frac{2n-1}{4n-1}$
Vacuum particles Neutrino ν	$\nu = \epsilon \otimes \pi$	$\nu = \epsilon \otimes \pi$	$n_\nu = \frac{1}{2}$	$\frac{2n-1/2}{4n-1} = \frac{1}{2}$	$\kappa_\nu = 0$
Plasma particles Neutron $\underline{n}$	$\epsilon \otimes \epsilon \otimes \pi \otimes \pi$	$-$	$n_{\underline{n}} = \frac{4n-1}{4n-1} = 1$	0	$\kappa_{\underline{n}} = 1$
Plasma particles electron/positron $e\,/\,p$	$e := \epsilon \otimes \epsilon$	$p := \pi \otimes \pi$	$n_e = \frac{2n-1}{4n-1}$	$n_p = \frac{2n}{4n-1}$	$\kappa_e = -\frac{1}{4n-1}$
Plasma particles electron/positron $p\,/\,e$	$p := \pi \otimes \pi$	$e := \epsilon \otimes \epsilon$	$n_p = \frac{2n}{4n-1}$	$n_e = \frac{2n-1}{4n-1}$	$\kappa_p = \frac{1}{4n-1}$
Mie particles Electroton $\underline{e}$	$\underline{e} := e \otimes \pi$ $\underline{e} = \epsilon \otimes \epsilon \otimes \pi$	π	$n_{\underline{e}} = \frac{3n-1}{4n-1}$	$\frac{n}{4n-1}$	$\kappa_{\underline{e}} = \frac{2n-1}{4n-1}$
Mie particles Magneton $\underline{m}$	$\underline{m} := p \otimes \epsilon$ $\underline{m} = \pi \otimes \pi \otimes \epsilon$	ϵ	$\underline{m} = \frac{3n-1/2}{4n-1}$	$\frac{n-1/2}{4n-1}$	$\kappa_{\underline{m}} = \frac{2n}{4n-1}$

Remark: The ranges of the considered sets of quantum numbers are

$$\kappa_\epsilon \in \left[-\frac{2}{3}, -\frac{1}{2}\right[,\ \kappa_\pi \in \left]-\frac{1}{2}, -\frac{1}{3}\right],\ \kappa_e \in \left[-\frac{1}{3}, -\frac{1}{4}\right[,\ \kappa_p \in \left]\frac{1}{4}, \frac{1}{3}\right],\ \kappa_{\underline{e}} \in \left[\frac{1}{3}, \frac{1}{2}\right[,\ \kappa_{\underline{m}} \in \left]\frac{1}{2}, \frac{2}{3}\right].$$

Remark (appendix): In the case, where the positive part of the spectrum of $W_{\kappa.\tau}$ lies in an interval $[m, b]$, where $m > 0$, then the inequality

$$\|W_{\kappa.\tau}x\|_{(\tau)} \geq \frac{m}{\sqrt{2}}\sqrt{\varphi^2_{\kappa.\tau}(x) + \|x\|^2_{(\tau)}} \geq \frac{m}{\sqrt{2}}\sqrt{c^2 + \|x\|^2_{(\tau)}}$$

holds for every x in the hyperbolic region V_c defined by $\sqrt{\varphi_{\kappa.\tau}(x)} \geq c > 0$, as well as in the conical region V_0, i.e., when $c = 0$.

4. Affected current mathematical realities

For more details related to this section we refer to (BrK1)

Primarily affected mathematical tools/equations

- The Mie equation is acompanied with the concept of an „electric force", counterbalanced in the ether by an „electric pressure"
  The problem of the field theory of matter as described by the Maxwell equations is, that they cannot explain (WeH) p. 170:, „why the field possesses a granular structure and why the knots of energy remain intact in spite of the back and forth flux of energy and momentum. The Maxwell equations will not do because they imply that negative charges compressed in an electron explode; to guarantee their coherence in spite of Coulomb's repulsive forces was the only service still required of the substance by H. A. Lorentz's theory of electrons. The preservation of the energy knots must result from the fact that the modified field laws admit only of one state of field equilibrium. … The target of the Mie theory is to explain, (WeH) p. 170: „*why the field possesses a granular structure and why the knots of energy remain intact in spite of the back and forth flux of energy and momentum* (*).
  In the proposed quantum potential model the „mass" of a quantum „particle" is modelled as an element of the „positive" sub-space of a Krein space accompanied by corresponding potential energy; its related „mass" is modelled by the discrete „energy knots" of the underlying kinetic potential operator. The Krein space based model supports the „preservation of the energy knots" requirement in spite of the back and forth flux of quantum energy governed by related angular operator (due to the compact embedding $H_{\kappa.\alpha}^{+} \subset H_{\kappa.(\tau)}^{+}$ enabled by the Krein space intrinsic angular operator

- Schrödinger's „only one problem" in statistical thermodynamics
  (ScE) p. 1: „*There is, essentially, only one problem in statistical thermodynamics: the distribution of a given amount of energy E over N identical systems. Or perhaps better: to determine the distribution of an assembly of N identical systems over the possible states in which this assembly can find itself, given that the energy of the assembly is a constant E*".
  In the proposed quantum potential model every considered „elementary particle" object is an element of a Hilbert-Krein space governed by a corresponding potential operator. All considered Hilbert-Krein sub-spaces can be related to the statistical Hilbert space L_2. Additonally, the proposed quantum potential model provides some evidence for the famous Schrödinger momentum operator $\boldsymbol{p} = -ih\nabla = -ih\partial$, also supporting his concept of negotropy in the context of „living systems", (ScE1)

- Dirac's theory of radiation of an electron
  E. Fermi: „Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field".
  In the proposed quantum potential model the potential operator becomes an intrinsic part of the Krein space model and not a physical phenomenon specific appropriately chosen function, like the Coulomb potential function in Dirac's theory of radiation. The kinetical energy system is defined by the energy knots of an underlying kinetic energy operator. In other words, the need for a small term representing the coupling energy of the atom and the radiation field disappears

- Heisenberg's proposed mathematical tool for an unified field theory
  in the context of the S-matrix and field operators for a mathematical description of free particles and the degeneracy of the ground state, (HeW)
  The proposed mathematical tool is an indefinite metric concept.

(*) (WeH) p. 170: „*since all physically important properties of an elementary material particle belong to the surrounding field rather than the substantial nucleus at the field center, the question becomes inevitable whether the existence of such a nucleus is not a presumption that may be completely dispensed with. … This question is answered in the affirmative by the field theory of matter. According to the latter a material particle such as an electron is merely a small domain of the electric field within which the field strength assumes enormously high values, indicating that a comparatively huge field energy is concentrated in a very small space. Such an energy knot, which by no means is clearly delineated against the remaining field, propagates through empty space like water waves across the surface of a lake; there is no such thing as one and the same substance of which the electron exists at all times. …. It (the field) is of the essence of the continuum. Even the atomic nuclei and the electrons are not ultimate unchangeable elements that are pushed back and forth by natural forces acting upon them, but they are themselves spread out continuously and are subject to fine fluent changes*".
(WeH1) p. 208: „*the ponderomotive force occuring in the Mie equation is contrasted with the „electrical force" E by $E - \nabla\varphi = 0$, that is the electric force E is counterbalanced in the ether by an electric pressure φ. In other words, the potential appears as an electric pressure; this is the required cohesive pressure that keeps the electron together*".

Affected Millennium Problems of the Clay Mathematics Institute

- *Yang–Mills and Mass Gap*
  Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known

  The physical „mass gap" problem of the three elementary particles $W^{\pm}$ and Z (whereby the charges of those particles have very similar properties with the positrons, electrons and photons) is due to the mathematical inconsistencies between all affected elementary particles.

  The proposed quantum potential model enabling the Mie equation model accompanied by the concept of discrete „energy knots" makes the YME obsolete

- *Navier–Stokes Equation*
  This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

  All attempts to prove well-posed 3D NSE faled due to inappropriate (not bounded) energy norm estimates

  The variational representation of the 3D-NSE governed by the least energy principle with respect to the potential energy norm $\|x\|_{1/2}^2$ ensures convergent energy norm estimates (appendix). Additionally, Mie's electric pressure concept applied to the second unknown function in the NSE, the „pressure p" enables a reformulation of the Navier-Stokes Equations $\dot{u} - \Delta u + (u \cdot \nabla)u = \nabla p$ where $p = \sum_{i,j=1}^{3} R_i R_j (u_i u_j)$ and ∇p becomes the Calderón-Zygmund integrodifferential operator (EsG) p. 44.

- *Riemann Hypothesis*
  The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

  (DeJ) p. 292: The Montgomery-Odlyzko law: "The distribution of the spacing between successive non-trivial zeros of the Riemann zeta function (suitable normalized) is statistically identical with the distribution of eigenvalue spacing in a GUE (Gaussian Unitary Ensemble) operator.
  (DeJ) p. 295: "What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?"

  We claim, that the plasma potential operator provides an appropriate model for the existence of the conjectured Hermitian „Berry-Keating" operator Its mathematical counterpart is the Hilbert-Polya conjecture, which is equivalent to the Riemann Hypothesis, (BrK1). For supporting formulae we refer to appendix A.

  With respect to the Goldbach conjecture we note that the proposed sets of quantum number systems was motivated by the proposed two-semicircle method to replace the Hardy-Littlewood circle method enabling a proof of the Goldbach conjecture, (BrK2), (BrK3).

Quadratic and complementary „least energy" Riesz-Galerkin methods

Hilbert-Krein space based least energy variational pseudo-differential equation representations enable the full power of quadratic and complementary „least energy" Riesz-Galerkin methods accompanied by FEM, BEM, and wavelet approximation methods, (BrK).

The construction of an operator algebra consisting of integral and differential operators leads to the concept of pseudo-differential operators. The PDO theory provides the appropriate framework for affected physical differential and (singular) integral equations. In order to apply „least energy" Riesz-Galerkin methods it requires strong elliptic pseudo-differential operators, (BrK1). The hyperbolic wave equation operator (the D'Alembert operator) with domain in a $H_{(\tau)}$ framework defines a strong hyperbolic pseudo-differential operators (BrK1). This allows to revisit the current concept of „wave front sets" of the standard pseudo-differential operator theory, (PeB).

5. Affected current physical realities

For more details related to this section we refer to (BrK1)

Microscopic and macroscopic Landau damping phenomena

Plasma is an inoized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. The nearly equal numbers of the plasma electron & positron elements is the most relevant physical differentiator between plasma matter states and „standard“ matter states.

(DeR) p. 94: „*The Landau damping phenomenon is complementary to the properties of electro-magnetic forces, which weaken themselves spontaneously over time w/o increase of entropy or friction. Landau damping involves a flow of energy between single particles on the one hand side, and collective excitations of plasma on the other side*". Physically speaking, there is a wave damping without energy dissipation by elementary particle collisions, i.e. the Landau damping phenomena is about the possibility of resonance between the wave phase velocity and the velocity of individual electrons.

The Landau damping phenomenon may be interpreted as

- a long term asymptotic approximation behavior to neutron elements
- the capability of stars to organize themselves in a stable arrangement in a vacuum (*).

The proposed model supports the above physical modelling requirement. Accordingly, a „wave equation“ model needs to govern two nearly equal electron & positron waves. Mathematically speaking, this leads to a governing telegraph equation model.

$\boldsymbol{\alpha, \beta, \gamma}$-decay phenomena

The α, β, γ-decay phenomena may be also called α, β, γ- „energic particle transformations“. In the proposed model those different energic particle transformations are governed by the table of affected sets of quantum numbers.

The β-decay is about an ionized radiation, which occurs by the β transformation of an energic β-particle of a nucleus (i.e., a „neutron“) leaving the nucleus, while generating at the same time an anti-neutrino or a neutrino. On average this neutron decays into an electron and a proton within 15 minutes.

The only „standard models“ of the SMEP, which do not annihilate each other are the electrons and the positrons. They emit photons in the form $e^+ + e^- \rightarrow \gamma\gamma$. In the proposed model the neutrino may be interpreted as a photon. It avoids the up/down attributes of the related „standard models“ of the SMEP accompanied by the YME model.

The α-decay is about instable atomic nuclei. The related energic particle transformations are the change from one atomic nucleus to another, both governed by different kinetic potential operators. The mathematical tool governing this process with respect to the impact of such a change on the corresponding potential operator changes is given by the inequalities

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{\tau(\delta^{-1} - \sqrt{\lambda})} \quad \text{for any } \tau, \delta, \alpha > 0 \text{ and } \lambda \geq 1$$

$$\vartheta^{-2\alpha} \leq \theta^{-2\alpha} + e^{\tau(\theta - \vartheta)} \quad \text{for any } \tau, \theta, \alpha > 0 \text{ and } \vartheta \geq 1.$$

(*) (ShF) p. 402: "*In its purest form, Landau damping represents a phase-space behavior peculiar to collisionless systems. Analogs to Landau damping exist, for example, in the interactions of stars in a galaxy at the Lindblad resonances of a spiral downsity wave. Such resonances in an inhomogeneous medium can produce wave absorption (in space rather than in time), which does not usually happen in fluid systems in the absence of dissipative forces (an exception in the behavior of corotation resonances for density waves in a gaseous medium) … Landau amplification can take place only for charged-particle distribution functions that display some sort of anomalous behavior in phase space*".

The Lamb shift and the fine structure constant

The Lamb shift phenomenon says that the energy values of an electron in the hydrogen potential field shows slighly different values than the related discrete energy knots $\sqrt{\lambda_n} = n$. The fine structure constant of the hydrogen atom is a mathematical correction term in the Sommerfeld energy formula derived from the „hydrogen“ Dirac equation governed by the Coulomb potential operator based on the „physical relevant“ stationary solutions of this equation. It is derived from Dirac‘s spin-orbit momentum operator by a mathematical „trick“, which may be called „two-component-separation of the angular momentum“. Beside the restriction to only physical relevant solutions of the Dirac equation there is also a mathematical approximation error, which is caused by a finite power series approximation of this separation (phys., a shrinking of the set of quantum numbers).

The proposed model provides a representation of the Hermitian operator as the sum of two hermitian kinetic and potential operators accompanied by the sum of two corresponding energy norms without any technical approximation errors caused by the *s*mall term representing the coupling energy of the atom and the radiation field in the Dirac electron model.

Other physical coupling constants

The Krein space framework provides the concept of „potential barriers“ between $x^+_{\kappa.(\tau)}$ and $x^-_{\kappa.(\tau)}$ quantum elements. For given constants c the potential of the elements $x^+_{\kappa.(\tau)} + x^-_{\kappa.(\tau)}$ enables the definition of corresponding hyperboloids H_c, hyperbolic regions V_c, conical regions V_0, and the boundary of conical regions, which are asymptotic conical manifolds for the hyperboloids.

The most prominent candidates for those potential constants in the plasma case are the the permeability constants ε_0 resp. μ_0 of elementary particles of an electric resp. magnetic field in a vacuum. As there are two „wave types“ affected the dynamics of both fields is governed by the telegraph equation.

The related „electric resp. magnetic mass“ constants are the natural candidates for the two field in the Mie theory. Accordingly, their corresponding dynamics is governed by two wave equations in line with the Maxwell equations. Accordingly, the today’s formula $\varepsilon_0 \cdot \mu_0 = \frac{1}{c^2}$ as a consequence of the two independent wave equations need to be revisited.

The Planck action constant requires the concept of „physical time“ accompanied by kinematic (time-dependent) energy norms.

Superfluids, Superconductors, and Bose-Einstein-Condensates

The Ginzburg-Landau theory in a magnetic field accompanied with the concept of surfaces of superconductors. In (GiJ) Ginzburg has presented the notion of Krein spaces that is an extension of Hilbert spaces for studying in quantum mechanics.

The BCS theory of superconductors is accompanied by the concepts of Cooper pairs and a mean-field-Hamilton operator. In the proposed model the Cooper pairs may be interpreted as alternating (maximal) pairs. The counterpart of the mean-field-Hamilton operator might become the plasma potential operator.

Ehrenhaft‘s photophoresis phenomenon

Ehrenhaft’s comment on his discovered phenomenon of photophoresis by electric and magnetic ions was :

> *„Particles of matter irradiated by light between electrodes behave as if they carry positive or negative electric charges. Therefore we can say that through the action of the light uncharged particles obtain unipolar charges, either negative or positive“.*
>
> *It is unlikely, that all those movement phenomena in light with or without the action of a field can be explained with the helf of today’s hypothesis; we may be forced to reach for something new.*

The proposed model explains this phenomenon from a theoretical perspective.

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Appendix A

The Hilbert spaces H_α, $H_{(\tau)}$, $H_\alpha \otimes H_{\alpha.(\tau)}$

For the technical details we refer to the appendix B. Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. Then Hilbert spaces $\{H_\alpha | \alpha \in R\}$ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \, , x_n := (x, \varphi_n).$$

In case of $\alpha = 0$ we skip the subscript.The bilinear form $a(x,y) := (Ax, y)$ defines an inner (kinetic energy) product in $D(A) = H_1$ and the operator equation $Ax = f$ is equivalent to, (BrK),

$$(x,y)_1 = (f, y), \forall y \in H_1.$$

For $\alpha < 0$ the Fourier coefficients x_n contribute to the α -norm with a polynomial decay. For $\tau > 0$ the inner product resp. norm in the form

$$(x,y)_{(\tau)} = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n y_n \, , \|x\|_{(\tau)}^2 = (x,x)_{(\tau)}$$

spanning the Hilbert space $H_{(\tau)}$ have an exponential decay with

$$\|x\|_{(\tau)}^2 \le c(\alpha,\tau) \|x\|_\alpha^2, \forall x \in H_\alpha.$$

The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(\tau)}^2 \text{ with } \alpha, \delta > 0 \text{ being arbitrary.}$$

Especially for $\alpha = 1/2$ one get

$$\|x\|_{-1/2}^2 \le \delta \|x\|_0^2 + e^{\tau/\delta} \|x\|_{(\tau)}^2 \text{ with } \delta > 0 \text{ being arbitrary.}$$

Putting

$$\|x\|_{\alpha.(\tau)}^2 := \sum_{n=1}^\infty \lambda_n^\alpha e^{-\sqrt{\lambda_n}\tau} x_n^2$$

one gets, (appendix, (BrK1))

i) $\int_0^\infty \|x\|_{(\tau)}^2 d\tau = \sum_{n=1}^\infty \lambda_n^{-1/2} x_n^2 = \|x\|_{-1/2}^2 \le \delta \|x\|_0^2 + e^{\tau/\delta} \|x\|_{(\tau)}^2$ for $\delta > 0$

ii) $(\ddot{x}, x)_{(\tau)} = \|\dot{x}\|_{(\tau)}^2 = \frac{1}{4} \sum_{n=1}^\infty \lambda_n e^{-\sqrt{\lambda_n}\tau} x_n^2 = \frac{1}{4} \|x\|_{1/2}^2$

iii) $\int_0^\infty \|\dot{x}\|_{(\tau)}^2 d\tau = \frac{1}{4} \sum_{n=1}^\infty x_n^2 = \frac{1}{4} \|x\|_0^2$.

Remark: We note that the D'Alembert operator with domain $L_2(H_{\alpha.(\tau)})$ is a strongly hyperbolic operator.

Let $\Phi_n := \varphi_n^H$ denote the Hilbert transform of φ_n with $(\varphi_n, \Phi_n) = 0$, (BrK1). The Hilbert space H_α of the composition $H_\alpha \otimes H_{\alpha.(\tau)}$ is built by the orthogonal system $\{\varphi_n\}$ while the Hilbert space $H_{(\tau)}$ is built by the orthogonal system $\{\Phi_n\}$ equipped with the related inner products resp. norms in the form

$$(x,y)_\alpha = \sum_{n=1}^{\infty} \lambda_n^\alpha x_n^{kin} y_n^{kin}, \qquad \|x\|_\alpha^2 = (x,x)_\alpha, \qquad x_n^{kin} := (x, \varphi_n), \;\; \alpha \in R$$

$$(x,y)_{\alpha.(\tau)} = \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-\sqrt{\lambda_n}\tau} x_n^{pot} y_n^{pot}, \quad \|x\|_{\alpha.(\tau)}^2 = (x,x)_{\alpha.(\tau)}, \qquad x_n^{pot} := (x, \Phi_n), \;\; \tau > 0.$$

In the following we shall omit the Fourier coefficient indices refering to the related *kin*etic and *pot*ential energy norm case.

Then, the system $\left\{\psi_{n.\alpha.\tau}^{(1)}, \psi_{n.\alpha.\tau}^{(2)}\right\}$ with

$$\psi_{n.\alpha.\tau}^{(1)} := \lambda_n^{\alpha/2} \varphi_n - i\lambda_n^{\alpha/2} \Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}, \quad \psi_{n.\alpha.\tau}^{(2)} := \lambda_n^{\alpha/2} \varphi_n + i\lambda_n^{\alpha/2} \Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}$$

defines an orthogonal system of the Hilbert space composition $H_\alpha \otimes H_{\alpha.(\tau)}$. For

$$x_{\alpha.\tau}^{(1)} := \sum_{n=1}^{\infty} x_n \psi_{n.\alpha.\tau}^{(1)}, \quad x_\tau^{(2)} := \sum_{n=1}^{\infty} x_n \psi_{n.\alpha.\tau}^{(2)}$$

the corresponding inner product of $H_\alpha \otimes H_{\alpha.(\tau)}$ is given by

$$\left(x_{\alpha.\tau}^{(1)}, x_{\alpha.\tau}^{(2)}\right) = (x,y)_\alpha + (x,y)_{\alpha.(\tau)}.$$

The relationship between the norms above and there relationship to the statistical L_2 norm is given by

$$\int_0^\infty \|x\|_{\alpha.(\tau)}^2 d\tau = \sum_{n=1}^{\infty} \lambda_n^\alpha \lambda_n^{-1/2} x_n^2 = \|x\|_{\alpha-1/2}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2 \text{ for } \delta > 0$$

which is a consequence from (appendix)

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2$$

with $\delta > 0$ being arbitrary.

The Krein space $\boldsymbol{H_{(\tau)}=H^{+}_{\kappa.(\tau)}\otimes H^{-}_{\kappa.(\tau)}}$

The Hilbert space $H_{(\tau)}$ decomposition in the form

$$H_{(\tau)}=H^{+}_{\kappa.(\tau)}\otimes H^{-}_{\kappa.(\tau)}$$

is supposed to be a quanta potential Hilbert-Krein space framework, where the parameter κ relates to correspondingly defined quantum number sequences in the form

$$\kappa^{+}_{\tau.n}:=\frac{1}{2}\frac{e^{\kappa_n\tau}}{\cosh(\kappa_n\tau)}\,,\ \kappa^{-}_{n}:=\frac{1}{2}\frac{e^{-\kappa_n\tau}}{\cosh(\kappa_n\tau)}\ \text{ with } \kappa_n\in R.$$

For

$$x_{(\tau)}:=\sum_{n=1}^{\infty}e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n\in H_{(\tau)}$$

$$x^{+}_{\kappa.(\tau)}:=\sum_{n=1}^{\infty}\kappa^{+}_{\tau.n}e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n\in H^{+}_{\kappa.(\tau)}$$

$$x^{-}_{\kappa.(\tau)}:=\sum_{n=1}^{\infty}\kappa^{-}_{\tau.n}e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n\in H^{-}_{\kappa.(\tau)}$$

it follows (*)

$$x_{(\tau)}=x^{+}_{\kappa.(\tau)}+x^{-}_{\kappa.(\tau)}\,.$$

The Hilbert space decomposition $H_{(\tau)}=H^{+}_{\kappa.(\tau)}\otimes H^{-}_{\kappa.(\tau)}$ is accompanied by the indefinite inner products resp. metric

$$[x,y]_{\kappa.(\tau)}:=\left(x^{+}_{\kappa.(\tau)},y^{+}_{\kappa.(\tau)}\right)-\left(x^{-}_{\kappa(\tau)},y^{-}_{\kappa.(\tau)}\right)$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sinh(2\kappa_n\tau)}{\cosh^2(\kappa_n\tau)}e^{-\sqrt{\lambda_n}\tau}x_ny_n$$

$$=\sum_{n=1}^{\infty}\tanh(\kappa_n\tau)e^{-\sqrt{\lambda_n}\tau}x_ny_n$$

where

$$\left(x^{+}_{\kappa.(\tau)},y^{+}_{\kappa.(\tau)}\right):=\sum_{n=1}^{\infty}(\kappa^{+}_{\tau.n})^2e^{-\sqrt{\lambda_n}\tau}x_ny_n$$

$$\left(x^{-}_{\kappa(\tau)},y^{-}_{\kappa.(\tau)}\right):=\sum_{n=1}^{\infty}(\kappa^{-}_{\tau.n})^2e^{-\sqrt{\lambda_n}\tau}x_ny_n\,.$$

We note the corresponding relations in the form

$$x^{+}_{\kappa.(\tau)}-x^{-}_{\kappa.(\tau)}=\sum_{n=1}^{\infty}\tanh(\kappa_n\tau)e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n$$

$$\left\|x^{+}_{\kappa.(\tau)}\right\|^2-\left\|x^{-}_{\kappa(\tau)}\right\|^2=\sum_{n=1}^{\infty}\tanh(\kappa_n\tau)\,e^{-\sqrt{\lambda_n}\tau}x_n^2=[x,x]_{\kappa.(\tau)}\,.$$

From the equivalent formulas

$$(x,y)_{(\tau)}=\left[x^{+}_{\kappa.(\tau)},y^{+}_{\kappa.(\tau)}\right]-\left[x^{-}_{\kappa.(\tau)},y^{-}_{\kappa.(\tau)}\right]$$

$$[x,y]_{\kappa.(\tau)}:=\left(x^{+}_{\kappa.(\tau)},y^{+}_{\kappa.(\tau)}\right)-\left(x^{-}_{\kappa(\tau)},y^{-}_{\kappa.(\tau)}\right)$$

it follows the characterization of „positive", „negative", and „neutral" vectors $x\in H_{(\tau)}$ by the relations

$$\left\|x^{+}_{\kappa.(\tau)}\right\|>\left\|x^{-}_{\kappa(\tau)}\right\|\ ,\ \left\|x^{+}_{\kappa.(\tau)}\right\|<\left\|x^{-}_{\kappa.(\tau)}\right\|\ ,\ \left\|x^{+}_{\kappa.(\tau)}\right\|>\left\|x^{-}_{\kappa(\tau)}\right\|\,.$$

(*) appendix: $\kappa_{\tau.n}^{+}+\kappa_{\tau.n}^{-}=1$, $\kappa_{\tau.n}^{+}-\kappa_{\tau.n}^{-}=\tanh(\kappa_n\tau)$, $(\kappa_{\tau.n}^{+})^2-(\kappa_{\tau.n}^{-})^2=\frac{sinh\,(2\kappa_n\tau)}{cosh^2(\kappa_n\tau)}=\tanh(\kappa_n\tau)$.

The potential operator of a Krein space

The canonical J-symmetric operator of a Krein space may be intepreted as a *„potential"* operator W (VaM) p. 90. In our case it is defined by

$$W_{\kappa.\tau}x:=\frac{1}{2}grad\varphi_{\kappa.\tau}(x):=x_{\kappa.(\tau)}^{+}-x_{\kappa.(\tau)}^{-}=\sum_{n=1}^{\infty}tanh(\kappa_n\tau)e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n\ .$$

It is complete, invertible, isometric ($W_{\kappa.\tau}=W_{\kappa.\tau}^{-1}$) and symmetric. Thus, the bilinear form

$$((x,y))_{\kappa.(\tau)}:=[W_{\kappa.\tau}x,y]_{\kappa.(\tau)}=\sum_{n=1}^{\infty}tanh^2(\kappa_n\tau)\,e^{-\sqrt{\lambda_n}\tau}x_ny_n$$

defines an inner product on all of the Hilbert space $H_{(\tau)}$ with related norm

$$|||x|||_{\kappa.(\tau)}^2:=[W_{\kappa.\tau}x,x]_{\kappa.(\tau)}=\sum_{n=1}^{\infty}tanh^2(\kappa_n\tau)\,e^{-\sqrt{\lambda_n}\tau}x_n^2\ .$$

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection" of the space $H_{(\tau)}$ in the subspace $H_{\kappa.(\tau)}^{+}$. The sub-space $H_{\kappa.(\tau)}^{+}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda=1$. The sub-space $H_{\kappa.(\tau)}^{-}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda=-1$. The whole spectrum of $W_{\kappa.\tau}$ lies on the join of the points $\lambda=\pm1$.

We note that the operator norm of the potential operator with respect to the inner product $(x,y)_{(\tau)}$ is equivalent to the $|||x|||_{\kappa.(\tau)}^2$, i.e., it holds

$$(W_{\kappa.\tau}x,W_{\kappa.\tau}y)_{(\tau)}=\sum_{n=1}^{\infty}tanh^2(\kappa_n\tau)\,e^{-\sqrt{\lambda_n}\tau}x_ny_n=((x,y))_{\kappa.(\tau)}\ .$$

The potential and hyperboloids of a Krein space

The indefinite metric (functional) of the considered Krein space

$$\varphi_{\kappa.\tau}(x):=[x,x]_{\kappa.(\tau)}=\left\|x_{\kappa.(\tau)}^{+}\right\|^2-\left\|x_{\kappa(\tau)}^{-}\right\|^2=\sum_{n=1}^{\infty}tanh\,(\kappa_n\tau)e^{-\sqrt{\lambda_n}\tau}x_n^2\ .$$

in combination with the functional $\big((x)\big):=\sqrt{\varphi_{\kappa.\tau}(x)}$ generates hyperboloids H_c, hyperbolic regions V_c, and conical region V_0 in the form

$$H_c:=\{x\in H_{(\tau)}|\varphi_{\kappa.\tau}(x)=c>0\},V_c:=\{x\in H_{(\tau)}|\big((x)\big)\geq c>0\},V_0:=\{x\in H_{(\tau)}|\big((x)\big)\geq 0\}.$$

Evidently V_c is a subspace of V_0.

(VaM) p. 91: *„If x is an exterior point of the conical region V_0, then those points of the ray $tx, t\in[0,\infty)$ for which $t\geq c/a$ belong to the hyperbolic region V_c, and those for which $0\leq t<c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t\in[0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big)=c>0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big)=0$. If we look at the unit sphere S^1 ($\|x\|^2=1$), then those points of S^1 for which $\left\|x_{\kappa.(\tau)}^{+}\right\|=\left\|x_{\kappa(\tau)}^{-}\right\|$ belong to K, and those points of S^1 for which $\left\|x_{\kappa.(\tau)}^{+}\right\|>\left\|x_{\kappa(\tau)}^{-}\right\|$ intersect the hyperboloid $\big((x)\big)=c>0$ at the point whose distance from θ is given by*

$$t=c(\left\|x_{\kappa.(\tau)}^{+}\right\|^2-\left\|x_{\kappa(\tau)}^{-}\right\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ *if* $\left\|x^{+}_{\kappa.(\tau)}\right\|^{2} - \left\|x^{-}_{\kappa(\tau)}\right\|^{2} \to 0$*, i.e. the manifold* K *is an asymptotic conical manifold for the hyperboloid* $\big((x)\big) = c > 0$.“

The angular and dissipative operators of a Krein space

The counterparts of W-norms $|||x|||^{2}_{\kappa.(\tau)} := [W_{\kappa.\tau}x, x]_{\kappa.(\tau)}$ with respect to the H_{α} Hilbert spaces norms $\|x\|^{2}_{\alpha} = \sum_{1}^{\infty} \lambda_{n}^{\alpha} x_{n}^{2} < \infty$ are given by the norms

$$|||x|||^{2}_{\alpha.\kappa} := \sum_{n=1}^{\infty} tanh^{2}(\kappa_{n}\tau)\, \lambda_{n}^{\alpha} x_{n}^{2} .$$

Let $L := H_{\alpha.\kappa} \subset H_{(\tau)} = H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$ and $P^{\pm}$ be the canonical projectors. Then the bounded linear operator

$$K^{+} = K^{+}_{\kappa.(\tau)} \coloneqq P^{-}(P^{+}|H_{\alpha.\kappa})^{-1} : P^{+}|H_{\alpha.\kappa} \to H^{-}_{\kappa.(\tau)}$$

is called the angular operator for $H_{\alpha.\kappa}$ with respect to $H^{+}_{\kappa.(\tau)}$. Then, the set of vectors of the sub-space

$$L := H_{\alpha.\kappa} := \{x^{+}_{\alpha.\kappa} + Kx^{+}_{\alpha.\kappa}\}_{x^{+} \in H^{+}_{\kappa.\alpha}}$$

gives the general form of all $H^{+}_{\kappa.\alpha} \subset H^{+}_{\kappa.(\tau)}$ of the Krein space $H = H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$.

Theorem 11.7 ((BoJ) p. 54, (PhR)): A subspace $L \subset H_{(\tau)} = H^{+}_{\kappa.(\tau)} \otimes H^{-}_{\kappa.(\tau)}$ is positive if and only if the angular operator K^{+} of L with respect to $H^{+}_{\kappa.(\tau)}$ exists and satisfies the condition

$$\left|\left|\left|K^{+}_{\kappa.(\tau)} x^{+}_{\kappa.(\tau)}\right|\right|\right|^{2}_{\kappa.(\tau)} \le \left|\left|\left|x^{+}_{\kappa.(\tau)}\right|\right|\right|^{2}_{\kappa.(\tau)} , x^{+}_{\kappa.(\tau)} \in D(K^{+}_{\kappa.(\tau)}).$$

In particular, positive definite subspaces are characterized by the property

$$\left|\left|\left|K^{+}_{\kappa.(\tau)} x^{+}_{\kappa.(\tau)}\right|\right|\right|^{2}_{\kappa.(\tau)} < \left|\left|\left|x^{+}_{\kappa.(\tau)}\right|\right|\right|^{2}_{\kappa.(\tau)}, x^{+}_{\kappa.(\tau)} \in D(K^{+}_{\kappa.(\tau)}), x^{+}_{\kappa.(\tau)} \neq 0,$$

and neutral subspaces by

$$\left|\left|\left|K^{+}_{\kappa.(\tau)} x^{+}_{\kappa.(\tau)}\right|\right|\right|^{2}_{\kappa.(\tau)} = \left|\left|\left|x^{+}_{\kappa.(\tau)}\right|\right|\right|^{2}_{\kappa.(\tau)}, x^{+}_{\kappa.(\tau)} \in D(K^{+}_{\kappa.(\tau)}).$$

The inclusion $H^{+}_{\kappa.\alpha} \subset H^{+}_{\kappa.(\tau)}$ is accompanied by related inclusions $H^{-}_{\kappa.\alpha} \subset H^{-}_{\kappa.(\tau)}$. The related Krein space concept is called alternating (maximal) pairs and alternating extensions.

The physical application of maximal positive and negative sub-spaces is concerned with the concept of of maximal dissipative (and maximal accretive) operators accompanied with spectra of unitary and self-adjoint operators, (BoJ) p. 114 ff.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_{1}^{(0)}, T_{2}^{(0)}$ of dissipative operators T_{1}, T_{2} with dense domains $D(L_{1})$, $\mathrm{D}(L_{2})$ in H_{0} (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_{1}x_{1}, x_{1}] + [x_{1}, T_{1}x_{1}] \le 0, x_{1} \in D(T_{1})$$

$$[T_{2}x_{2}, x_{2}] + [x_{2}, T_{2}x_{2}] \le 0, x_{2} \in D(T_{2}).$$

Mean ergotic theorem: If U is an isometry on a complex Hilbert space and if P is a projection on the space of all vectors invariant under U, then $\frac{1}{n}\sum_{j=0}^{n-1} U^{j}x$ converges to Px for every x in the space.

Strong elliptic and hyperbolic PDO
(BrK1)

By construction the Hilbert scales characterized by a polynomial decay in case of λ_i^α enables optimal shift theorem for the Laplacian operator in the form, (appendix I)

$$\|x\|_{\alpha+2}^2 = (Ax, Ax)_\alpha = \|Ax\|_\alpha^2 .$$

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert (wave) operator related to the wave equation:

$$\Box w := \ddot{w} - \Delta w \ .$$

The Hilbert space defined by the inner product resp. norm

$$(x,y)_{(t)}^2 = \sum_{i=1}^\infty e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \ \ t > 0$$

$$\|x\|_{(t)}^2 = (x,x)_{(t)}^2$$

provides „optimal“ shift theorems for related strong hyperbolic operators.

Theorem: For the D'Alembert (wave) operator it holds

$$\int_0^T \|w\|_{k+2,(t)}^2 dt \le c \int_0^T \|f\|_{k,(t)}^2 dt \ .$$

Proof: Let $w_i := (w, \phi_i)$ resp. $f_i := (f, \phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-w_i'' = \lambda_i w_i$ of the Laplacian operator. Th corresponding solution of ($\Box w = f$),

$$\ddot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \ \text{ and } w_i(0) = \dot{w}_i(0) = 0 \ .$$

is given by

$$w_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau\right) f_i(\tau) d\tau .$$

It holds for $\tau \le t$

$$\int_0^T \|w\|_{k+2,(t)}^2 dt = \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} w_i^2(t) dt$$

$$\le \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau\right) f_i(\tau) d\tau\right]^2 dt$$

$$\le \sum \lambda_i^{k+1} \int_0^T e^{-\sqrt{\lambda_i}t} \left(\int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau\right) d\tau\right) \left[\int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau\right) d\tau \, f_i^2(\tau) d\tau\right] dt$$

$$\le \sum \lambda_i^{k+1/2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\int_0^t f_i^2(\tau) d\tau\right] dt \ .$$

Exchanging the order of integration gives

$$\int_0^T \int_0^t e^{-\sqrt{\lambda_i}t} f_i^2(\tau) d\tau dt = \int_0^T \int_t^T e^{-\sqrt{\lambda_i}t} f_i^2(\tau) dt d\tau$$

$$= \int_0^T f_i^2(\tau) dt \left[\int_t^T e^{-\sqrt{\lambda_i}t} d\tau\right]$$

$$\le \frac{1}{\sqrt{\lambda_i}} \int_0^T f_i^2(\tau) dt \ .$$

Theorem: In general there exists no „optimal" hyperbolic shift theorem the standard Sobolev space framework in the form

$$\|w\|_{k+2}^{2} \leq c\|f\|_{k}^{2}$$

Proof: the counter example is given by the function

$$\Phi(x,t) := e^{-(\frac{1}{2}-(x-t))^{2}}, u(x,t) := t^{2}\Phi(x,t), f(x,t) := 2\Phi(x,t) - 4t\Phi'(x,t)$$

fulfilling the relationships

$$\dot{\Phi}(x,t) = -\Phi'(x,t), \ddot{\Phi}(x,t) = \Phi''(x,t), \ddot{u}(x,t) - u''(x,t) = f(x,t)$$

and

$$\|u''\|_{L_2(L_2)} \sim \|\Phi''\|_{L_2(L_2)} \quad \text{but} \quad \|f\|_{L_2(L_2)} \sim \|\Phi'\|_{L_2(L_2)}.$$

Global boundedness of the 3D-Navier-Stokes equations
in a variational $H_{-1/2}$ based Hilbert space framework
(BrK1)

It turned out that based on the physical modelling assumption of a variational representation of the 3D NSE in a $H_{-1/2}$ Hilbert space framework (interpreted as a fluid element test space) the 3D NSE enjoy global solutions. Its a consequence of the well-known Sobolevskii-estimates for the 3D case. Those estimates fail in case of a H_0 test space.

Lemma (Sobolevskii): For $0 \le \delta < 1/2 + n \cdot (1 - 1/p)/2$ it holds

$$\left|A^{-\delta} P(u, grad)v\right|_p \le M \cdot \left|A^{\theta} u\right|_p \cdot |A^{\rho} u|_p$$

with a constant $M := M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \ge n/2p + 1/2, \theta, \rho > 0, \theta + \rho > 1/2$.

The NSE initial-boundary equation is given by

$$\frac{du}{dt} + Au + Bu = Pf \;, u(0) = u_0$$

where $B(u) := P(u, grad)u$ and $Pu_0 = u_0$. Multiplying this homogeneous equation with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_{\alpha} + (Au, u)_{\alpha} + (Bu, u)_{\alpha} = 0, \; (u(0), v)_{\alpha} = (u_0, v)_{\alpha} \text{ for all } v \in H_{-1/2}.$$

For $\alpha := -1/2$ one gets the generalized "energy" inequality in the form

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

Putting $p := 2$, $\delta := 1/4$,$\theta := \rho := 1/2$ fulfilling $\theta + \rho \ge \frac{1}{4}(n+1) = 1$ one gets from the Sobolevskii-lemma above

$$\left\|A^{-\delta}P(u, grad)u\right\| \le c\left\|A^{\theta}u\right\| \cdot \|A^{\rho}u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\},$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

Remark: We note that the pressure p in the variational representation

$$(Au, v)_{-\frac{1}{2}} \coloneqq (\nabla u, \nabla v)_{-\frac{1}{2}} + (\nabla p, v)_{-\frac{1}{2}} = (u, v)_{\frac{1}{2}} + (p, v)_0 \qquad \text{for all } v \in H_{-1/2}$$

$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2}$$

can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

with (R_1, R_2, R_3) is the Riesz transform.

Some formulas
(Grl)

i) $cosh(x) \pm sinh(x) = e^{\pm x}$

ii) $cosh^2(x) - sinh^2(x) = 1$

iii) $tanh(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx}$, $x > 0$ (Grl) 1.232

iv) $tanh(x) = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)}{(2k)!}B_{2k}x^{2k-1}$

v) $e^{ax} - e^{bx} = (a-b)xe^{\frac{1}{2}(a+b)x}\prod_{k=1}^{\infty}(1 + \frac{(a-b)^2x^2}{4k^2\pi^2}$ (Grl) 1.223

vi) $sinh(2x) = 2\,sinh(x)\,cosh(x)$, $cosh(2x) = 2cosh^2(x) - 1$ (Grl) 1.334

vii) $tanh(x)\frac{sinh\,(2x)}{cosh^2(x)} = tanh(x)\frac{2\,sinh(x)cosh(x)}{cosh^2(x)} = 2\,tanh^2(x)$

viii) $\int sinh(ax)dx = \frac{1}{a}cosh(ax)$, $\int cosh(ax)dx = \frac{1}{a}sinh(ax)$ (Grl) 2.414

ix) $\int\frac{dx}{cosh^2(x)} = tanh(x)$, $\int\frac{dx}{sinh^2(x)} = -coth(x)$ (Grl) 2.422

x) $\int tanh(x)dx = ln(cosh(x))$, $\int coth(x)dx = ln(sinh(x))$ (Grl) 2.423

xi) $\int\frac{\sinh\,(2nx)}{\cosh\,(x)}dx = 2\sum_{k=0}^{n-1}(-1)^k\frac{\cosh((2n-2k-1)x)}{2n-2k-1}$ (Grl) 2.433

xii) $\int\frac{sinh\,(2x)}{cosh\,(x)}dx = 2cosh(x)$

xiii) $\int tanh(x)dx = ln(cosh(x))$, $\int coth(x)dx = ln(sinh(x))$ (Grl) 2.423

xiv) $\int_0^{\infty} e^{-\alpha x}\,tanh(x)\,dx = \beta\left(\frac{\alpha}{2}\right) - \frac{1}{\alpha}$, $Re(\alpha) > 0$ (Grl) 3.541

xv) $a^2 \neq b^2$, (Grl) 2.481,

$$\int e^{ax}sinh\,(bx+c)dx = \frac{e^{ax}}{a^2-b^2}[a\cdot sinh\,(bx+c) - b\cdot cosh(bx+c)]$$

$$\int e^{ax}cosh\,(bx+c)dx = \frac{e^{ax}}{a^2-b^2}[a\cdot cosh\,(bx+c) - b\cdot sinh(bx+c)]$$

xvi) $a^2 \neq b^2$, (Grl) 2.484,

$$\int e^{ax}sinh\,(bx)\frac{dx}{x} = \frac{1}{2}\{[Ei(a+b)x] - [Ei(a-b)x]\}$$

$$\int e^{ax}cosh\,(bx)\frac{dx}{x} = \frac{1}{2}\{[Ei(a+b)x] + [Ei(a-b)x]\}$$

xvii) $tanh(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx}$ (Grl) 1.232

xviii) $e^{ax} - e^{bx} = (a-b)xe^{\frac{1}{2}(a+b)x}\prod_{k=1}^{\infty}(1 + \frac{(a-b)^2x^2}{4k^2\pi^2}$ (Grl) 1.223

xix) $\int_0^\infty e^{-zx}\tanh(x)\,dx = \beta\left(\frac{z}{2}\right) - \frac{1}{z}\,, \beta(1) = log2, \beta\left(\frac{1}{2}\right) = \frac{\pi}{2}\,,\ Re(z) > 0$ (Grl) 3.541

Appendix B

For the cited references we refer to (BrK1)

Appendix I

The Hilbert scale $\{\boldsymbol{H_\alpha}|\boldsymbol{\alpha} \in \boldsymbol{R}\}$

(BrK1)

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$ and the norm $\|..\|$ Let A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A).$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

there is a countable sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_i = \lambda_i\phi_i \ ,\ (\phi_i, \phi_k) = \delta_{i,k} \quad \text{and} \quad \lim_{i\to\infty}\lambda_i \to \infty$$

any $x \in H$ is represented by

$$(*) \quad x = \sum_{i=1}^{\infty}(x, \phi_i)\,\phi_i \ \text{ and } \ \|x\|^2 = \sum_1^\infty(x, \phi_i)^2.$$

Similarly one can define the spaces H_α, where the case $\alpha < 0$ is related to the theory of distributions. They consist of those elements $x \in H$ with scalar product

$$(x, y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x, \phi_i)\,(y, \phi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (*) it holds

$$\|x\|_2^2 = (Ax, Ax)_0\,, H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma :

i) Let $\alpha < \beta$. Then $\|x\|_\alpha \leq \|x\|_\beta$ for $x \in H_\beta$ and the embedding $H_\beta \to H_\alpha$ is compact.

ii) Let $\alpha < \beta < \gamma$. Then $\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu$ for $x \in H_\gamma$ with $\mu = \frac{\gamma-\beta}{\gamma-\alpha}$ and $\nu = \frac{\beta-\alpha}{\gamma-\alpha}$.

iii) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

iv) $\|x - y\|_\alpha \leq t^{\beta-\alpha} \|x\|_\beta$

v) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

vi) $\|y\|_\gamma \leq t^{-(\gamma-\beta)} \|x\|_\beta$

Lemma:

i) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

ii) $\|x - y\|_\rho \leq t^{\beta-\rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

iii) $\|y\|_\sigma \leq t^{-(\sigma-\beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

The extended Hilbert space $H_{\alpha.(\tau)}$
(BrK1)

The extended Hilbert space $H_{\alpha.(\tau)}$ is defined by the following inner product resp. norm

$$(x,y)_{(\tau)} = \sum_{i=1} e^{-\sqrt{\lambda_i}\tau}(x,\phi_i)(y,\phi_i), \quad \|x\|_{(\tau)}^2 = (x,x)_{(\tau)}.$$

The (τ)-norm is weaker than any α-norm, i.e.

$$\|x\|_{(\tau)}^2 \leq c\|x\|_\alpha^2 \quad \text{for any } \alpha\text{-norm}$$

with $c = c(\alpha,\tau)$ depending only on α and τ.

The counterpart of the related lemmata of the considered Hilbert scale is

Lemma: Let $\tau, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_\tau(x)$ according to

$$\|x - y\| \leq \|x\|$$

$$\|y\|_1 \leq \delta^{-1}\|x\|$$

$$\|x - y\|_{(\tau)} \leq e^{-\tau/\delta}\|x\|.$$

Any Hilbert scale norm with negative index, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (τ)-norm:

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2$$

with $\delta > 0$ being arbitrary.

Proof: The inequality is a consequence of the following inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{\tau(\delta^{-1}-\sqrt{\lambda})}, \text{ for any } \tau,\delta,\alpha > 0 \text{ and } \lambda \geq 1.$$

If $\lambda^{-1/2} \leq \delta$ then obviously $\lambda^{-\alpha} \leq \delta^{2\alpha}$, in case of $\lambda^{-1/2} \geq \delta$ it holds $e^{\tau(\delta^{-1}-\sqrt{\lambda})} \geq 1$, whereas $\lambda^{-\alpha} \leq 1$ is a consequence of $\alpha > 0$ and $\lambda \geq 1$.

Putting $\delta = \frac{1}{\theta}$ and $\lambda = \vartheta^2 \geq 1$ it follows from the lemma above the

Corollary: for any $\tau, \theta, \alpha > 0$ and $\vartheta \geq 1$ the following inequality is valid

$$\vartheta^{-2\alpha} \leq \theta^{-2\alpha} + e^{\tau(\theta-\vartheta)}.$$

Lemma: Because of $\int_0^\infty e^{-\sqrt{\lambda_i}\tau}d\tau = \frac{1}{\sqrt{\lambda_i}}$ it holds

$$\int_0^\infty \|x\|_{(\tau)}^2 d\tau = \|x\|_{-1/2}^2.$$

Appendix II

Krein spaces, potentials and potential operators

(AzT), (BoJ)

A Krein space is a Hilbert space H with inner product (x,y), which can be written in the form $H = H^+ \otimes H^-$. There are two equivalent approaches defining Krein spaces based on

the concept of an indefinite metric (also called a Q-metric) $Q(x,y) \coloneqq [x,y]$, $\forall x,y \in H$

a self-adjoint operator B defined on all of the Hilbert space H inducing the decomposition of H.

A canonical decomposition of $H = H^+ + H^-$ enables the (positive definite) inner product of H according to

$$(*)\ \ (x,y) = [x^+,y^+] - [x^-,y^-], x = x^+ + x^-, y = y^+ + y^-.$$

For vectors $u,v \in H^+$ we have $(u,v) = [u,v]$; for vectors $u,v \in H^-$ we have $(u,v) = -[u,v]$. If $u \in H^+$ and $v \in H^-$, then it follows from (*) that $(u,v) = [u,,\theta] - [,\theta,v]$.

The formula (*) can be inverted in the following way

$$[x,y] = (x^+,y^+) - (x^-,y^-) \text{ resp. } [x,x] = (x^+,x^+) - (x^-,x^-)$$

from which it follows

„Positivity, negativity, and neutrality of a vector $x \in H$ are equivalent to the relations

$\|x^+\| > \|x^-\|$, $\|x^+\| < \|x^-\|$, or $\|x^+\| > \|x^-\|$ respectively.“

In short, a Krein space can be looked on as an arbitrary Hilbert space decomposed into usual orthogonal sums of two subspaces, equipped in addition to the original Hilbert metric (i.e., the inner product (x,y)) with an additional indefinite metric $[x,y]$.

The decomposition of a Krein space generates two mutually complementary projectors P^+ and P^- mapping H on to H^+ and H^- respectively. Those orthogonal projection operators P^+ and P^- are linked to the indefinite metric by, (VaM) chapter IV,

$$\varphi(x) := [x,x] = \|P^+x\|^2 - \|P^-x\|^2 .$$

The indefinite metric $\varphi(x)$ can be interpreted as a „potential“. The related „potential operator“ (in mathematics it is called „the canonical symmetry“ J, (AzT) §3, (BoJ) p. 52) is then given by, (VaM) (10.7), (12.6)

$$\boldsymbol{W}(x) := \frac{1}{2} grad\varphi(x) = P^+x - P^-x = x^+ - x^-.$$

The fundamental properties of the potential operator $\boldsymbol{W}(x)$ are completeness, invertibility, ($\boldsymbol{W} = \boldsymbol{W}^{-1}$) isometry, and symmetry. Thus, the bilinear form $(x,y)_W \coloneqq (\boldsymbol{W}(x),y)$ defines an inner product, (BoJ) p. 52.

The sub-space H^+ is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = 1$.

The sub-space H^- is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = -1$.

The whole spectrum of $\boldsymbol{W}$ lies on the join of the points $\lambda = \pm 1$.

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection“ of the space H in the subspace H^+ .

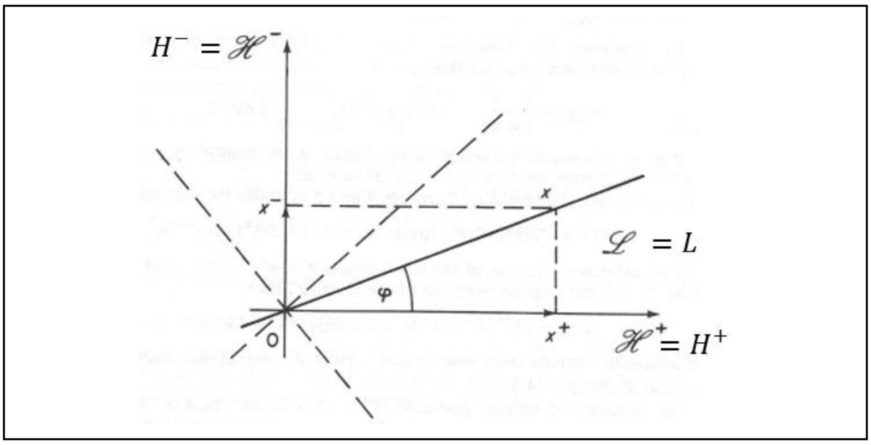
Krein spaces and angular (dissipative and accretive) operators
(AzT), (BoJ)

By the aid of J-norms a description of semi-definite subspaces L can be given enabling the definition of an angular operator $K^+: H^+ \to H^-$ with domain $D(K^+) = P^+(L)$ and range $R(K^+) = P^-(L)$, (BoJ) p. 54. For the following we refer to (AzT) p. 48 ff. and (BoJ) p. 54.

Let $L \subset H$ in a Krein space $H = H^+ \otimes H^-$ and $P^\pm$ the canonical projectors (. Then the bounded linear operator

$$K^+ := P^-(P^+|L)^{-1} : P^+|L \to H^-$$

is called the angular operator for L with respect to H^+. The meaning of this nomenclature is explained by the following picture, (AzT) p. 61:



In the figure above a non-negative (even positive) subspace $L \subset H$ is shown. For any $x \in L$ we have $x = x^+ - x^-$, and $x^- = Kx^+$, where K is the operator of rotating the vector x^+ through an angle $\pi/2$ (in the positive direction), and then multiplying by a scalar $k = tan\varphi$ - the angular coefficient of the „line" L . ...

If φ is always understood to be the *minimal* angle between L and „the axis" H^+, then $\tan(\varphi) = \|K\|$. In the general case too ($dimH \leq \infty$) for the angular operator K of a non-negative subspace L we have $tan\,(\varphi(L, H^+) = \|K\|$, if the (minimal) angle φ is defined by the equality $sin\,(\varphi(L, H^+) = sup_{e \in S(L)}\|e - ZP^+e\|$, where $S(L)$ is the unit sphere of the lineal L ($\|e\| = 1$).

Theorem 8.2 ((AzT) p. 49; see also Theorem 11.6, (BoJ) p. 54): The set of vectors

$$L = \{x^+ + Kx^+\}_{x^+ \in L^+}$$

in which L^+ is an arbitrary lineal from H^+, and $K : L^+ \to H^-$ is an arbitrary compression ($\|K\| \leq 1$), gives the general form of all $L \subset H$ of the Krein space $H = H^+ \otimes H^-$, and $L^+ = P^+(L)$ and K is the angular operator for L with respect to H^+.

Let $\|x\|_W^2 = \|x\|_J^2 = \|x^+\|^2 - \|x^-\|^2$ denote the $J = W$-inner product related (potential) norm.

Theorem 11.7 ((BoJ) p. 54): A subspace $L \subset H$ is positive if and only if the angular operator K^+ of L with respect to H^+ exists and satisfies the condition

$$\|K^+x^+\|_W^2 \leq \|x^+\|_W^2 \,, x^+ \in D(K^+).$$

In particular, positive definite subspaces are characterized by the property

$$\|K^+x^+\|_W^2 < \|x^+\|_W^2 \,, x^+ \in D(K^+), x^+ \neq 0,$$

and neutral subspaces by

$$\|K^+x^+\|_W^2 = \|x^+\|_W^2 \,, x^+ \in D(K^+).$$

For negative subspaces similar statements, involving K^- instead of K^+, are valid.
Theorem 8.2' ((AzT) p. 49): The set of vectors

$$L = \{Qx^- + x^-\}_{x^- \in L^-}$$

in which L^- is an arbitrary lineal from H^-, and $Q : L^- \to H^+$ is an arbitrary compression ($\|Q\| < 1$), gives the general form of all $L^+ \subset H$ of the Krein space $H = H^+ \otimes H^-$, and $L^- = P^-(L)$ and Q is the angular operator for L with respect to H^-.

Alternating pairs and dissipative operators in Hilbert space
(BoJ)

(BoJ) p. 39: Let H_0 denote a Hilbert space with with inner product $(x,y)_0$, $x,y \in H_0$ and norm $\|x\|$ and let W be an arbitrary bounded self-adjoint operator ($W = W^*$) given on H_0. Then the Hermitian sesquilinear form $[x,y] = (Wx,y)_0 = Q(x,y)$ defines in H_0 an indefinite metric which we shall call the W-metric, and we shall call the space H_0 itself with the W-metric a W-space. W is called the Gram operator of the space H_0.

(BoJ) p. 91: A linear operator A with an arbitrary domain of definition $D(A)$, operating in a W-space H_0, is said to be W-dissipative if $Im[Ax,x] \geq 0$ for all $x \in D(A)$, and to be maximal W-dissipative if it is W-dissipative and coincides with any W -dissipative extension of it.

An ordered pair of subspaces $\{L_1, L_2\}$ of the Krein space H will be called an alternating pair provided L_1 is positive, L_2 is negative, and $L_1 \perp L_2$. If, in addition, L_1 is maximal positive and L_2 is maximal negative, the pair $\{L_1, L_2\}$ is called alternating maximal pair.

By an alternating extension of the alternating pair $\{L_1, L_2\}$ we mean an alternating pair $\{L_1' L_2'\}$ such that $L_1 \subset L_1'$, $L_2 \subset L_2'$.

Theorem 9.1 (BoJ) p. 115: Every alternating pair in the Krein space H can be extended to an alternating maximal pair.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $D(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1x_1, x_1] + [x_1, T_1x_1] \leq 0,\ x_1 \in D(T_1)$$

$$[T_2x_2, x_2] + [x_2, T_2x_2] \leq 0,\ x_2 \in D(T_2).$$

Theorem (BoJ) p. 118: If $\left\{L_1^{(0)}, L_1^{(0)}\right\}$ is an alternating maximal pair extending $\{D(-T_1), D(-T_2)\}$, then the operators $T_1^{(0)}, T_2^{(0)}$ defined by the relations $L_1^{(0)} = D(-T_1^{(0)})$, $L_2^{(0)} = D(-T_2^{(0)})$ are maximal dissipative operators of the dissipative operators T_1, T_2, and every solution can be obtained in this way.

Krein spaces and hyperboloids
accompanied by hyperbolic and conical regions
(VaM) p. 89 ff.

Putting $x^+ := P^+x, x^- := P^-x$ the corresponding potential $\varphi(x^+ + x^-)$ defined by

$$\varphi(x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0$$

generates hyperboloids in the form

$$H_c := \{x \in H | (x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0\}.$$

A hyperbolic region V_c is defined by

$$\big((x)\big) = \sqrt{\|x^+\|^2 - \|x^-\|^2} \geq c > 0$$

A conical region V_0 is defined by

$$\big((x)\big) = \sqrt{\|x^+\|^2 - \|x^-\|^2} \geq 0 \, .$$

Evidently V_c is a subspace of V_0.

If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0, \infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0, \infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 \ = 1$), then those points of S^1 for which $\|P^+x\| = \|P^-x\|$ belong to K, and those points of S^1 for which $\|P^+x\| > \|P^-x\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by

$$t = \mathrm{c}(\|x^+\|^2 - \|x^-\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ if $\|x^+\|^2 - \|x^-\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$.

Lemma: If the (proper) subspace $H_1 \subset H$ is finite dimensional, then the region V_c ($c \geq 0$) is weakly closed.

Remark: Ellipsoids are defined by the condition $\frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} = 1$. The related elliptical region is defined by

$$E_c := \left\{x \in H | \frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} \leq c, c > 0\right\}.$$

Theorem (ZaC) p. 291: Let H denote a Hilbert space with inner product $(\cdot,\cdot)$ and $K \subset H$ be a closed convex cone. For every $x \in H$ let $P^K x$ (which is uniqly defined) denote the projection of x on K. Putting $K^- := -K^+ := \{y \in H] \ (x, y) \leq 0, \forall x \in H\}$ it holds $\ \forall x \in H \ x = P^K x + P^{K^-} x$ and $(P^K x, P^{K^-} x) = 0$. Conversely, if $x = x_1 + x_2$ with $x_1 \in K, x_2 \in K^-$ and $(x_1, x_2) = 0$ then $x_1 = P^K x$ and $x_2 = P^{K^-} x$.

Hyperboloids generated by operators
(VaM) p. 92

Let B be self-adjoint operator defined on all of the Hilbert space H. Since it follows that B is bounded, then

$$inf\{\,(Bx,x)=a \mid \|x\|=1\}>\infty\,,\, sup\{\,(Bx,x)=b \mid \|x\|=1\}<\infty\ .$$

We shall assume that $a<0$, $b>0$. Further, let E_t be the resolution of the odentity corresponding to B; then $E_b-E_0=P_1$ is a projection operator onto subspace $H_1\subset H$ which reduces B. Thus, the operator induces a decomposition of into the direct sum of subspaces H_1 and H_2 ($H=H_1\otimes H_2$) and thereby generated a hyperboloid

$$\varphi(x)=\varphi(x^{+}+x^{-})=\sqrt{\|P_1\|^2-\|P_2\|^2}=c>0\,,$$

where P_2 is the projection onto H_2.

In this case where the positive part of the spectrum of B lies in an interval $[m,b]$, where $m>0$, then the inequality

$$\|Bx\|\geq\frac{m}{\sqrt{2}}\sqrt{\varphi^2(x)+\|x\|^2}\geq\frac{m}{\sqrt{2}}\sqrt{c^2+\|x\|^2}$$

holds for every x in the hyperbolic region V_c defined by

$$\varphi(x)=\sqrt{\|P^{+}x\|^2-\|P^{-}x\|^2}\geq c>0\,,$$

as well as in the conical region V_0 defined by

$$\varphi(x)=\sqrt{\|P^{+}x\|^2-\|P^{-}x\|^2}\geq 0\,.$$

Remark: It should be remarked that in some cases the operator B leaves invariant the hyperbolic regions V_c, which it generates. This is the case, for example, when the positive part of the spectrum of B lies in the interval $[1,b]$ and the negative part lies in $[-1,0]$. In fact, we then have

$$\big((Bx)\big)=\|P^{+}Bx\|^2-\|P^{-}Bx\|^2=\|BP^{+}x\|^2-\|BP^{-}x\|^2$$

$$=\int_1^b t^2d(E_t\,P^{+}x,P^{+}x)-\int_{-1}^0 t^2d(E_t\,P^{-}x,P^{-}x)$$

$$\geq\|P^{+}x\|^2-\|P^{-}x\|^2\geq c^2.$$

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QUANTUM THEORY OF RADIATION*

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TABLE OF CONTENTS

Part I. Dirac's Theory of Radiation

§1. Fundamental concept

Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representing the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field.

If we neglect this last term, the atom and the field could not affect each other in any way; that is, no radiation energy could be either emitted or absorbed by the atom. A very simple example will explain these relations. Let us consider a pendulum which corresponds to the atom, and an oscillating string in the neighborhood of the pendulum which represents the radiation field. If there is no connection between the pendulum and the string, the two systems vibrate quite independently of each other; the energy is in this case simply the sum of the energy of the pendulum and the energy of the string with no interaction term. To obtain a mechanical representation of this term, let us tie the mass M of the pendulum to a point A of the string by means of a very thin and elastic thread a. The effect of this thread is to perturb slightly the motion of the string and of the pendulum. Let us suppose for instance that at the time $t=0$, the string is in vibration and the pendulum is at rest. Through the elastic thread a the oscillating string transmits to the pendulum very slight forces having the same periods as the vibrations of the string. If these periods are different from the period of the pendulum, the amplitude of its vibrations remains always exceedingly small; but if a period of the string is equal to the period of the pendulum, there is resonance and the amplitude of vibration of the pendulum becomes considerable after a certain time. This process corresponds to the absorption of radiation by the atom.

If we suppose, on the contrary, that at the time $t=0$ the pendulum is oscillating and the string is at rest, the inverse phenomenon occurs. The forces transmitted through the elastic thread from the pendulum to the string put the string in vibration; but only the harmonics of the string, whose frequencies are very near the frequency of the pendulum reach a considerable amplitude. This process corresponds to the emission of radiation by the atom.