

A new ground state energy model

Dr. Klaus Braun
79106 Freiburg i. Br.
August 18, 2013

Abstract Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(\mathbb{R}^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a 2π -periodic function and \oint denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(\mathbb{R}^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{2\pi} \oint u(x) e^{-i\nu x} dx$$

enable the definitions of the norms (e.g. [ILi] 11.1.5, [KBr0])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 \cdot$$

We propose to replace current re-normalization techniques to overcome certain convergence issue concerning today's ground state energy model by a modified (less regular) Hilbert space framework than current $H := L_2^*(\Gamma)$ Hilbert space, based on the Hilbert space $H_{-1} = H_0 \otimes H_0^\perp$. The orthogonal projection from $H_{-1} \rightarrow H_0$ ensures consistency with today's standard L_2 -model.

The mathematical framework and the notation are given in [KBr3]. It is built on the (Pseudo-differential) model operators

$$(S_{-1}u)(x) := -\oint \log 2 \sin \frac{x-y}{2} u(y) dy =: Au(x) \quad , \quad (S_0u)(x) := \oint \frac{1}{2} \cot \frac{x-y}{2} u(y) dy =: Hu(x)$$

$$(S_1u)(x) := \oint \frac{1}{4 \sin^2 \frac{x-y}{2}} u(y) dy$$

The Dirichlet integral $D(u, v) := (u', v')$ defines the inner product of the "standard" "energy space". The proposed alternative model of a potential of J. Plemelj ([JPI] §8) in combination with the Hilbert transform operator S_0 in the form

$$(S_1u)(x) = \oint \frac{1}{2} \cot \frac{x-y}{2} d_1 u$$

is applied, to define an alternative inner product to the Dirichlet integral for the infinitesimal small quantum world in the form

$$(u, v)_E := ((du, dv)) := (S_1u, S_1v)_{-1} \quad , \quad u, v \in D(S_1) = H_0 \cdot$$

The generalizations of the model operators S_0, S_1 lead to the Riesz operators R_k and the Calderón-Zygmund operator ([GEs] (3.17), (3.35)), given by

$$(\Lambda u)(x) = \left(\sum_{k=1}^n R_k D_k u \right)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^n p.k. \int \sum_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy$$

In general, in the proposed Hilbert space framework with inner product $(\Lambda u, \Lambda v)$ the reverse Legendre transformation

$$d(g) = y d\psi - \frac{\partial f}{\partial x} dx + (dy dy)$$

is no longer valid. Therefore, in general the Hamiltonian and the Lagrangian formalisms are no longer equivalent, i.e. while the concept of "energy" of a mass element dm ([JPI] p. 12) in the form $\|dm\|$ is a still valid definition in the sense of above, but the concept of "force" is no longer be defined in corresponding (quantum mechanics) models for elements of the extended domain of the operator Λ .

Notations

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a 2π -periodic function and \oint denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real ν the Fourier coefficients

$$u_\nu := \frac{1}{2\pi} \oint u(x) e^{-i\nu x} dx$$

enable the definitions of the norms (see e.g. [ILi] Remark 11.1.5, [KBr0])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 .$$

There is a natural representation of the Fourier decomposition

$$u(x) = \frac{a_0}{2} + \sum_1^{\infty} a_\nu \cos(\nu x) + \sum_1^{\infty} b_\nu \sin(\nu x) := \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \in L_2$$

as Laurent series description in terms of a complex variable, defined on a circle $z = e^{ix}$:

$$u(z) := \tilde{u}(z) := u(x) = \sum_{-\infty}^{\infty} u_\nu z^\nu \in H := L_2^*(\Gamma) .$$

with

$$u_0 := \frac{a_0}{2} , \quad u_\nu := \frac{1}{2}(a_\nu - ib_\nu) , \quad c_{-\nu} := \frac{1}{2}(a_\nu + ib_\nu) , \quad \nu > 0 .$$

Then H is the space of L_2 -periodic function in R .

For $t = e^{i\vartheta}$, $t_0 = e^{i\vartheta_0}$ it holds

$$\frac{dt}{t-t_0} = \frac{1}{2} \cot \frac{\vartheta - \vartheta_0}{2} d\vartheta + \frac{i}{2} d\vartheta = \frac{1}{2} \cot \frac{\vartheta - \vartheta_0}{2} d\vartheta + \frac{1}{2} \frac{dt}{t} .$$

Remark: From [DGA] pp.63 and [SGr] 1.441, we recall

$$\frac{1}{2\pi} \oint_{0 \rightarrow 2\pi} \begin{Bmatrix} \sin n\vartheta \\ \cos n\vartheta \end{Bmatrix} \cot \frac{\vartheta - \vartheta_0}{2} d\vartheta = \begin{Bmatrix} -\cos(n\vartheta_0) \\ \sin(n\vartheta_0) \end{Bmatrix} , \quad \frac{1}{2\pi} \oint_{0 \rightarrow 2\pi} \cot \frac{\vartheta - \vartheta_0}{2} d\vartheta = 0$$

resp.

$$\frac{1}{2\pi} \oint_{0 \rightarrow 2\pi} e^{in\vartheta} \cot \frac{\vartheta - \vartheta_0}{2} d\vartheta = \begin{cases} -ie^{in\vartheta_0} & n = 1, 2, 3, \dots \\ 0 & n = 0 \\ ie^{in\vartheta_0} & n = -1, -2, \dots \end{cases}$$

From [ILi] (1.2.34) (see also [TGr]) we note the identity with a hyper singular integral equation of kernel of Hilbert type

$$-n(a_n \cos nx_0 + b_n \sin nx_0) = \frac{1}{4\pi} \int_0^{2\pi} \frac{a_n \cos nx + b_n \sin nx}{\sin^2 \frac{x_0 - x}{2}} dx .$$

This identity is related to the following integral operators ([ILi] (1.2.31)-(1.2.33), [Ili1])

$$(A) \quad (Au)(x) := -\int \log 2 \sin \frac{x-y}{2} u(y) dy =: \int k(x-y) u(y) dy \quad \text{and} \quad D(A) = H = L_2^*(\Gamma)$$

$$(H) \quad (Hu)(x) := [u](x) := \frac{1}{2} \int \cot \frac{x-y}{2} u(y) dy = -\lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\varepsilon}^1 [u(x+y) - u(x-y)] \cot \frac{y}{2} dy \quad .$$

the following properties are valid:

Lemma

i) The operator H is skew symmetric in the space $L_2(0, 2\pi)$ (e.g. [DGA], [CRu], [BPe], [FTr] 4.2) and maps the space $H := L_2(0, 2\pi) - R$ isometric onto itself, and it holds

$$\|Hu\| = \|u\| \quad \text{and} \quad H^2 = -I \quad , \quad (Hu, v) = -(u, Hv) \quad , \quad H[u'](x) = [Hu]'(x)$$

$$(Hu)_v = -\text{sign}(v) u_v \quad , \quad (Hu)(x) = i \sum_1^{\infty} [u_{-v} e^{-ivx} - u_v e^{ivx}] \in L_2 \quad \text{for} \quad u \in L_2$$

It further holds:

i) The Fourier term for $v = 0$ is $(Hu)_0 = 0$, ([EPe] 2.9)

$$ii) \quad H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z))}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z) dz$$

$$\text{and therefore} \quad H(xu(x)) = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$$

iii) for odd functions it hold $H(xu(x)) = x(Hu)(x)$

$$iv) \quad Hu(x) = u(x) * \frac{1}{\pi x} \quad , \quad \frac{1}{\pi x} = \lim_{\rho \rightarrow 0} \frac{x}{\pi(x^2 + \rho^2)} \quad , \quad ([EPe] 2.9)$$

v) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e.

$$\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = 0 \quad ,$$

because of

$$\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = \frac{i}{2\pi} \int_{-\infty}^{\infty} \text{sign}(\omega) |\hat{u}(\omega)|^2 d\omega \quad \text{with} \quad |\hat{u}(\omega)|^2 \text{ is even.}$$

iii) The operator A is symmetric in its domain $D(A)$ and the Fourier coefficients the convolution integral is given by

$$(Au)_v = k_v u_v = \frac{1}{2|v|} u_v \quad , \quad D(A) \subseteq H_A = H_{-1/2}(\Gamma) \quad .$$

J. Plemelj's suggestion ([JPI] XV, p. 12, p. 17), see also [JAh], [JNi]) is about a relationship between the differential form calculus and its application in physics (e.g. [HCa], [HF1]) and a modified representation of the potential in the form

$$(*) \quad v(s) = -\frac{1}{\pi} \oint \log|\zeta(s) - \zeta(t)|u(t)dt \quad \text{by} \quad (**) \quad v(s) = -\frac{1}{\pi} \oint \log|\zeta(s) - \zeta(t)|du(t)$$

Plemelj's quote: *“Bisher war es üblich, für das Potential die Form (*) zu nehmen. Eine solche Einschränkung erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (**) verwendbar.“*

J. Plemelj ([JPI]) stated that the standard definition of the normal derivative is (just) not useful. He alternatively proposed $\oint u_n ds$, which he called “current”. The definition requires no “existing” boundary value of the derivatives, its defined only by the behavior/regularity of the function in its interior domain. With respect to the calculation of a potential this leads to a replacement of the concept of a mass density $\mu'(s)ds$ by a mass element $d\mu_s$ defined on an infinitely small piece of the boundary. It only requires

$$\int_{\sigma}^s |d\mu| \approx |s - \sigma|^{\rho} \quad \text{with} \quad \rho > 0 .$$

In the plane the “current” with respect to such a potential u is equivalent to the alteration of its conjugate potential $\bar{u} = H[u]$ between the infinitesimal small distances of the two end points on the boundary.

The proposed new “energy” inner product: The Dirichlet integral

$$D(u, v) := (u', v') \quad , \quad u, v \in H_1$$

defines the inner product of the “standard” “energy space”. We apply the concept of J. Plemelj to the Hilbert transform operator S_0 in the form

$$(S_1 u)(x) = \int \frac{1}{2} \cot \frac{x-y}{2} d_y u$$

in order to define a Dirichlet integral like inner product (with alternative Hilbert space domain, which requires less regularity assumptions than $\in H_1$) by

$$\mathbf{(E(1))} \quad (u, v)_E := ((du, dv)) := (S_1 u, S_1 v)_{-1}, \quad u, v \in D(S_1) = H_0 \quad .$$

The “extension” of the Hilbert transform for $n > 1$ is given by the Riesz transforms ([BPe], [EST] (3.35), [EST1] III, 4.2, IV, 6.4). Those transforms enable the corresponding definition of the Dirichlet integral “extension” for $n > 1$ in the form

$$\mathbf{(E(n))} \quad \left(\begin{pmatrix} du_1 \\ du_2 \\ \dots \\ du_n \end{pmatrix}, \begin{pmatrix} dv_1 \\ dv_2 \\ \dots \\ dv_n \end{pmatrix} \right) := \sum_{j=1}^n (R_j(du_j), R_j(dv_j))_{-1}$$

The operator S_1 is a Calderón-Zygmund integrodifferential operator with symbol $|v|$ ([GEs] (3.17), (3.35)), i.e. of the form

$$(\Lambda u)(x) = \sum_{k=1}^n R_k D_k u(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^n p.v. \int \sum_{-\infty}^{\infty} \frac{x_k - y_k}{|x - y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n-1}{2}}} p.v. \int \frac{\Delta_y u(y)}{|x - y|^{n-1}} dy = -(\Delta \Lambda^{-1})u(x)$$

whereby R_k denotes the Riesz operators ([HAb] p. 19, 106, [BPe] example 9.9, 9.10)

$$R_k u = -i \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} p.v. \int \frac{x_k - y_k}{|x - y|^{n+1}} u(y) dy$$

and it holds ([GEs] (3.15))

$$\Lambda^{-1} u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}} \int \frac{u(y) dy}{|x - y|^{n-1}} \quad , \quad n \geq 2 \quad .$$

Remark: We note that the gradient $\frac{\partial \Phi_k}{\partial x_k}$ is the prototype of co-variant vector fields and that

dx_k is the (infinitely small) contra-variant vector field ([ESc1] chapter 1) In case of an infinitely small parallel displacement there should be no change to the length of the vector. Displacement of coordinate differentials, when moving from $x_k \rightarrow x_k + dx_k$ enables the definition of a measure ds of the length of an infinitely small “distance” and $\int ds$ a measure of the length of a finite small “distance” ([ESc1] chapter VII: “The geodesics of an affine connexion”, see also [JPI]).

Remark: For corresponding physical models (e.g. Navier-Stokes equations, Maxwell equations and Einstein's field equations) to apply variational calculus in combination with this kind of inner product for differential forms the physical science we refer to e.g. [HFI], [ESc], [RSe].

We mention a few model problems/examples:

Prandtl's airfoil uplift theory (e.g. [LPr], [HSc], [FTTr] 4.3), based on the concepts of a "lifting line", "circulations", "Newton and Non-Newton fluid", leads to an integro-differential equation in the form

$$G(y) = L(y) \left[\alpha(y) - \frac{1}{2\pi} p.v. \int_{-1}^1 \frac{G'(\eta)}{y - \eta} d\eta \right]$$

With less regularity assumptions to its solution $G(y)$ this can be reformulated as singular integral (fix point) operator equation in the form

$$G(y) = L(y) [\alpha(y) + cS_1[G](y)].$$

The **pressure potential formulation of the Stokes flow** ([PCo], [JHe], [HLo],[DSh], [SMo], [CRu]) is the Pressure Poisson Equation (PPE) in the form

$$(*) \quad \Delta p = \nabla \cdot (f - (u \cdot \nabla)u) \quad \text{resp. a weak form like} \quad D(p, v) = (\nabla \cdot ((u \cdot \nabla)u) - f, v) \quad \text{for } v \in H^*$$

It applies in the context of incompressible Navier-Stokes equation

$$(1) \quad u_t + (u \cdot \nabla)u = \mu \Delta u - \nabla p + f$$

$$(2) \quad \nabla \cdot u = 0$$

in combination with the boundary conditions

$$(3) \quad u = g(x, t) \quad \text{for } x \in \partial\Omega$$

$$\text{where} \quad \int_{\partial\Omega} g_n dA = 0 \quad .$$

From [DSh] 1.2.3, 2.2b, we cite the following:

"...Given the flow u , can the PPE be used to obtain the pressure? Again, at first sight, the answer to this question appears to be yes. After all, () is a Poisson equation for p , which should determine it uniquely – in case appropriate boundary conditions are given. The problem is: what boundary conditions? Evaluation of (1) at the boundary, with use of (3), shows that the flow velocity determines the whole gradient of the pressure at the boundary, which is too much for (*). Further, if only a portion of these boundary conditions are enforced when solving (*) – say, the normal component of (1) at the boundary, then how can one be sure that the whole of (1) applies at the boundary?"*

From [HLo] we cite:

„..... In der vorliegenden Arbeit werde ich zeigen, dass keine Stetigkeitseigenschaften (oder auch nur Summierbarkeitseigenschaften) der partiellen Ableitungen (der Cauchy-Riemannschen Differentialgleichungen) vorausgesetzt zu werden brauchen. Es genügt vielmehr, ihre blosse Existenz (und die Gültigkeit der Cauchy-Riemannschen Differentialgleichungen) im Bereich anzunehmen.“

In [JNi1] a direct proof of an unusual shift theorem for the Stokes flow in the $\|\cdot\|_{-1}$ norm is given.

Heisenberg uncertainty relation ([WHe], [ESc]): It states that the location and momentum of a particle cannot have same measurable results, i.e. the corresponding operators

$$\bar{x}\psi(x) := x\psi(x) \quad , \quad \bar{d}\psi(x) := \bar{p}\psi(x) := \frac{d}{dx}\psi(x) = \psi'(x)$$

do not have any common eigenvalue. This is because of the identity

$$(*) \quad [\bar{d}\bar{x} - \bar{x}\bar{d}]\psi(x) = \psi(x)$$

i.e. because of the fact, that the commutator $[\bar{d}\bar{x} - \bar{x}\bar{d}]$ is not identical to the zero operator. With the above and also referring to [MBe] we propose to apply the operators $S_{-1} := A, S_0 := H, S_1$ ([KBr3]) to enable a *common*, less regular domain of the location and momentum operators, which is again a Hilbert space with *common* eigenpairs. Nevertheless the Heisenberg uncertainty relation keeps still valid, when the new domain is projected to the original classical or variational domains of \bar{x} and \bar{p} , which are e.g. the Sobolev spaces L_2 and H_1 . The alternative operator definitions could be

$$\begin{aligned} \bar{x}\psi(x) \rightarrow x\psi(x) \quad , \quad \bar{d}\psi \rightarrow S_1\psi & \quad \text{for } \psi \in L_2, \\ \bar{x}\psi \rightarrow S_{-1}[x\psi] = A[x\psi] \quad , \quad \bar{d}\psi \rightarrow S_{-1}S_1\psi \cong S_0\psi \cong H\psi & \quad \text{for } \psi \in H_{-1}. \end{aligned}$$

From the above properties of the Hilbert transform we note that for the Hilbert transform commutator $\bar{H}v := [xH - Hx]v := [xS_0 - S_0x]v$ it holds

$$[xH - Hx]v(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} v(y)dy \quad .$$

As the constant Fourier term of a Hilbert transform $w = \bar{v} := Hv$ is vanishing, this results into

$$[xH - Hx]w = 0 \quad \text{for } w, Hv, v \in L_2 \quad .$$

Obviously it holds

$$(v, w) = 0 \quad \text{for } v \in L_2 \quad , \quad \bar{H}Hv + H\bar{H}v = 0 \quad \text{for } v \in L_2 \quad .$$

The **Schrödinger field equation** for the electrons wave functions $\psi(\vec{x}, t)$ reflects in the right way the experimental verified relationship between the group velocity and the wave number. The wave functions themselves do not have any physical meaning. But the intensities of fields, as e.g. (from Maxwell theory) the energy density and the Poynting vector or (from quantum mechanics) the Hamiltonian operator of a free string ([HRo2] (2.10.43))

$$H := \frac{1}{2\rho} P_0^2 + \sum_1^\infty \frac{1}{2\rho} P_n^2 + \frac{\rho}{2l} \omega_n^2 Q_n^2 = \frac{1}{2\rho} P_0^2 + \frac{1}{2} \sum_1^\infty \hbar \omega_n (A_n^* A_n + A_n A_n^*) = \frac{1}{2\rho} P_0^2 + \sum_1^\infty \hbar \omega_n A_n^* A_n + \frac{1}{2} \sum_1^\infty \hbar \omega_n ,$$

are modeled as squares of field quantities. We note that the series

$$E_0 := \frac{1}{2} \sum_1^\infty \hbar \omega_n$$

is divergent (!! in standard Hilbert space framework. In alignment with the “square concept” above, the current *interpretation* is ([HRo1] (1.6.16)), that the quantity

$$\rho(x, t) := |\psi(\vec{x}, t)|^2$$

models the density of the matter field of electrons. Based on this interpretation the continuity equations (which is the Schrödinger equation) is given by

$$\dot{\rho} + \operatorname{div} \left(\frac{\hbar}{2mi} \psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right) = 0 .$$

The same argumentation keeps valid in the new proposed framework.

The **Hermite polynomials** $\{\varphi_n\}$ form an orthogonal basis of the L_2 – Hilbert space (e.g. [HRo1] (2.10.31)), which is based on the above also true for $\{\psi_n\} := \{\bar{\varphi}_n\} := \{H\varphi_n\}$, fulfilling $(\varphi_n, \psi_n) = 0$ and $\bar{H}\psi_n$ (see appendix).

The Hilbert spaces $H_{-1/2}, H_{-1}$ are characterized by

$$H_{-1/2} = \left\{ \psi \mid \|\psi\|_{-1/2}^2 = (A\psi, \psi)_0 < \infty \right\} \quad , \quad H_{-1} = \left\{ \psi \mid \|\psi\|_{-1}^2 = (A\psi, A\psi)_0 < \infty \right\} .$$

We note that

$$H_{-1} = H_0 \otimes H_0^\perp .$$

For the Dirac function it holds

$$\delta(x) := \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} dk = \frac{1}{\pi} \int_0^\pi \cos(kx) dk \in H_{-n/2-\varepsilon} \subset H_{-1} \quad \text{for } n = 1 .$$

Let

$$s_1(x) := \frac{1}{4} \sin^{-2} \frac{x}{2} \quad (\text{whereby } \hat{s}_1(v) = -v \operatorname{sgn}(v))$$

and

$$s_2(x) := x s_1(x) \cdot$$

Let further

$$\bar{S}_1 \psi := [x S_1 - S_1 x] \psi = \int (s_2(x-y) \psi(y) dy \approx \int d\psi_y \cdot$$

Proposition: As alternative to the above (Lie bracket) commutator (*) we propose the (singular integral convolution) operator

$$\bar{S}_1 \psi := [x S_1 - S_1 x] \psi \quad , \quad D(S_1) = H_{-1} \cdot$$

Proposition: The mathematical relationship between and the concept of “force” (modeled by the Lagrangian formalism) and the concept of “energy” (modeled by the Hamiltonian formalism) is given by the Legendre transformation, defined by

$$g := g(x, y) := \psi y - f = y \psi(x, y) - f(x, y)$$

and its reverse operation

$$d(g) = y d\psi - \frac{\partial f}{\partial x} dx + (d\psi dy) \cdot$$

In case f' is well defined, the Hamiltonian and the Lagrangian formalism are equivalent. As a consequence, in the proposed new Hilbert space framework above, where on df is required to be an element of the domain of the Hilbert transform operator, the concept of force the above is no longer valid, resp. only then “existing”, when f' exists. According to J. P Lemelj’s quote above ([JPI]) this “dispossess the potential of its biggest efficiency”.

Hilbert transforms in Yukawan Potential theory: from [RDu] we cite:

“If H denotes the classical Hilbert transform and $Hu(x) = v(x)$, then the functions $u(x)$ and $v(x)$ are the values on the real axis of a pair of conjugate functions, harmonic in the upper half-plane. This note gives a generalization of the above concepts in which the Laplace equation $\Delta u = 0$ is replaced by the Yukawa equation $\Delta u = \mu^2 u$ and in which the Cauchy-Riemann equations have a corresponding generalization. This leads to a generalized Hilbert transform H_μ . The kernel functions of this new transform is expressible in terms of Bessel functions K_0 . The transform is of convolution type.”

Lie bracket and exterior derivative ([HWe], [SLi]): A differential form is a tensor field. Therefore its Lie derivative can be built enabled by a vector field. The Lie derivative of a differential form can be described by the exterior (Cartan) derivative. The Lie derivative of a differential form is again a differential form. The Lie derivative of a vector field Y with a vector field X is given by the Lie bracket of X and Y. Therefore, knowing the exterior derivative of a 1-form is the same as knowing the Lie bracket on a vector fields (see e.g. [SDo] 1.2).

The basic idea of S. Lie is about an appropriate replacement of finite (algebraic) group by a corresponding infinite (analysis) group concept, i.e. to extend from discrete (distances) to continuous (infinitely small, dx) symmetry transformations. With respect to the proposed energy inner product (built on differential forms dx) we recall from [SLi] p. 2 (see also [HWe1] p. 35, [HWe2] §19) the scope and the main result of the book, which built the foundation of the Lie algebra theory:

Scope ([SLi] p. 1): *„Definiert man den Inbegriff aller Bewegungen des Raumes durch analytische Gleichungen, so erhält man die Gleichungen einer Transformationsgruppe, welche sich von allen anderen Gruppen durch gewisse charakteristische Eigenschaften unterscheidet. Man kann sich nun die Aufgabe stellen, solche möglichst einfache Eigenschaften der besprochenen Gruppe zu finden, welche für sie charakteristisch ist.“*

Main result ([SLi] p. 2): *„Die Bewegungen des dreifach ausgedehnten Raumes bilden ein Gruppe von reellen Transformationen, welche die folgende Eigenschaft besitzt: Wird ein reeller Punkt und ein reelles hindurchgehendes Linienelement festgehalten, so ist immer noch kontinuierliche Bewegung möglich; wird jedoch ausserdem ein durch das Linienelement gehendes reelles Flächenelement festgehalten, so bleiben alle Punkte des Raumes in Ruhe.“*

Translation:

Scope ([SLi] p. 1): *“If one defines the concept of all possible displacements/movements by analytical equations, one gets the equations of a transformation group, which differs from other groups by specific characteristic properties. One can now take the task, to look for most easy properties of this group, which are characteristically for it.”*

Main result ([SLi] p. 2): *“*

The displacements/movements of the three dimensional space define a group of real valued transformations, which fulfill the following properties: if one fixes a real point and a real valued line element through it, then there is always a continuously movement possible; however, if one fixes additionally a real valued area element through the line element, then for all points of the space there is no movement possible.”

As a consequence the proposed energy inner product (E(n)) enables continuous energy flow only by line element. Already by fixed area element the corresponding energy level change requires to be discrete.

The Hodge theorem ([HFI] 8.4): Let ω be any p -form and η any $(p+1)$ -form. Putting $\delta\omega := (-1)^{np+n+1} * d * \omega$ then $(d\omega, \eta) = (\omega, \delta\eta)$ and there is a $(p-1)$ -form α , a $(p+1)$ -form β and a harmonic p -form γ such that

$$\omega = d\alpha + \delta\beta + \gamma .$$

The forms $d\alpha, \delta\beta, \gamma$ are unique.

For the relationship between differential forms and mathematical concepts like differential forms of geodesic curvature, total curvature and parallel transport, as well as the calculation of the total curvature of a surface by means of the first fundamental form we refer to e.g. [HCa]. For the method of Pfaffians in the theory of curves and surfaces in the context of conformal mapping and minimal surfaces we refer e.g. to [DSt].

The Hilbert-Einstein functional: As an application of nonlinear variation of an “energy” functional we note the variation of total curvature of the Hilbert-Einstein functional:

Let (M, g) be a compact Riemann manifold with volume element $dV_g = \sqrt{\det g_{ij}} dx^1 \wedge dx^2 \dots \wedge dx^n$ with varying metric g . The volume and total curvature (Hilbert-Einstein) functionals of g are given by

$$Vol(g) := \int \int_M dV_g \quad , \quad S(g) := \int \int_M S_g dV_g$$

Let f, h be symmetric $(0,2)$ -tensor, (e_1, e_2, \dots, e_n) an orthogonal basis with respect to g , $g_t = g + t \cdot h$ be a variation of the metric g and S_t the scalar curvature of g_t with $S = S_0$.

Putting

$$\langle f, h \rangle_g := \int \sum_{i,j} f(e_i, e_j) h(e_i, e_j) dV_g$$

it holds with the Ricci tensor Ric

$$\left. \frac{d}{dt} \right|_{t=0} S(g_t) = \left. \frac{d}{dt} \right|_{t=0} \int S_t dV_t = \left\langle \frac{S}{2} g - Ric, h \right\rangle_g .$$

In contrast to the Ricci tensor the tensor $Sg/2 - Ric$ is divergence free, which is also the case for $\Lambda g + Sg/2 - Ric$ with Einstein’s very small cosmological constant Λ , which he introduced to enable a static universe model. The proposed ground state energy for “objects” dm with its corresponding (“energy”) inner product might provide an additional rational for such a constant Λ .

We note, that the Einstein field equations, which state that the matter, described by the energy-momentum tensor is generated by the curvature of the space-time, is an *AXIOM*.

For the equivalence of extreme problems for nonlinear problems, built on symmetric bilinear form and convex functionals and corresponding variational calculus built on Gateaux differentials for nonlinear problems we refer to e.g. [WVe].

For nonlinear functional analysis we refer to e.g. [KDe], [MRu].

From ([ESc1] p.2, see also [HWe2] §19) we recall: “The geometric structure of the space-time model envisaged in the 1915 theory is embedded in the following two principles:

- (i) *Equivalence of all four-dimensional systems of coordinates obtained from any one of them by arbitrary (point-) transformation; (principle of general covariance/invariance)*
- (ii) *The continuum has a metrical connexion impressed on it: that is, at every point a certain quadratic form of the coordinate-differentials,*

$$g_{ik} dx_i dx_k$$

called the “square of the interval” between the two points in question, has a fundamental meaning, invariant in the aforesaid transformations.

This two principles are of different standing:

- (i) *... incarnates the idea of General Relativity ...*
- (ii) *... On the other hand, to adopt a metrical connexion straight away does not seem to be the simplest way of getting at it eventually, even if nothing more were intended than an exposition of the 1915 theory.*

... We shall therefore investigate the geometry of our continuum in three steps or stages, viz.

- (1) *When only general invariance (co-variance) is imposed;*
- (2) *When in addition an affine connexion is imposed;*
- (3) **When this is specialized** *to carry a metric.*

The concept of an affine connexion was introduced by H. Weyl ([HWe2]). We recall from [HWe1] p. 46:

“An die Stelle der von Helmholtz geforderten Homogenität des metrischen Feldes ist die Möglichkeit getreten, im Rahmen der feststehenden Natur der Metrik das metrische Feld beliebigen virtuellen Veränderungen zu unterwerfen. ... Welche quantitative Ausgestaltung auch immer im Rahmen der Natur der Metrik das metrische Feld gefunden haben mag, stets determiniert das metrische Feld eindeutig den affinen Zusammenhang. ... Wenn wir zeigen können, dass die in der wirklichen Welt herrschende Natur der Metrik die einzige ist, welche diesem Prinzip Genüge leistet, so haben wir wohl ein Recht zu der Behauptung, dass wir von dem neuen Gesichtspunkt aus, der ein den Kräften der Materie gegenüber nachgiebiges metrisches Feld annimmt, das Raumproblem befriedigend und vollständig gelöst haben. Seine Lösung ist mir erst vor etwa einem Jahre gelungen ...

... aus dem metrischen Zusammenhang er gibt sich also, die Drehgruppe in P sich von der u im unendlich benachbarten Punkte P_0 nur durch die Orientierung unterscheidet. ... und wenn wir stetig vom Punkte P_0 zu einem beliebigen Punkte der Mannigfaltigkeit übergehen, so erkennen wir, dass die Drehungsgruppen in allen Punkten der Mannigfaltigkeit von der gleichen Art sind, ... Das ist die ein- für allemal feste Natur der Metrik.“

The standard analysis in the 4-dimensional Einstein continuum requires concepts of adequate derivatives and integrals based on the physical assumption of general co-variance. It leads to the concept of tensor and scalar density, co-variant derivative and affine connexion ([ESc1], [HWe2]).

Some prototypes of corresponding operators are the gradient, the rotation, the divergence and the energy-momentum (matter) tensor T_{ik} ([ESc1] chapter II). The tensor T_{ik} and its related Riemann tensor R_{ik} are not symmetric ([ESc1] (8.9)) in contrast to the energy-momentum tensor for a particle of rest-mass

$$m \frac{dx_i}{ds} \frac{dx_k}{ds}$$

of the Restricted Theory of Relativity. The new concept above provides an opportunity for a gravity theory based on the principle of *general contra-variance*, than on *general co-variance*. This would basically require the definition of a contra-variant derivative, which creates out of (k,l) – tensor a $(k+1,l)$ – tensor, building on the $((du, dv))$ inner product. An analogue definition to the “standard” metric (enabling potentially a linear Lorentz transformation?, as for the Dirac equation in QED) might look like

$$g(dV, dW) := g^{\mu\nu} dV_\mu dW_\nu =: dV_\mu dW_\nu = ((dV, dW)) .$$

For the related properties of the Riesz operators with respect to commutation with translations and homothesis, as well as properties relative to rotations we refer to [BPe] example 9.9, 9.10 and [ESt]. With respect to the “relative to rotation” property we recall from [ESt]:

let

$$m := m(x) := (m_1(x), \dots, m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then

$$m(\rho(x)) = \rho(m(x))$$

whereby

$$R_k u = -i c_n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x - y|^{n+1}} u(y) dy \quad \text{with} \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

$$m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

And

$$\begin{aligned} m(\rho(x)) &= c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} \text{sign}(x\rho^{-1}(y)) + \log \left| \frac{1}{x\rho^{-1}(y)} \right| \right) \frac{y}{|y|} d\sigma(y) \\ &= c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} \text{sign}(xy) + \log \left| \frac{1}{xy} \right| \right) \frac{y}{|y|} d\sigma(y) . \end{aligned}$$

From [BPe] we note the following further properties of the Riesz operators: If $j \neq k$ then $R_j R_k$ is a singular convolution operator. On the other hand $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. It holds

$$\|R_j\| = 1 \quad , \quad R_j^* = -R_j \quad , \quad \sum R_j^2 = -I \quad , \quad \sum \|R_j u\|^2 = \|u\|^2 \quad , \quad u \in L_2 .$$

The Hardy spaces H^p : From [ESt1] IV 6.3, we note that a periodic function on R in the form

$$u(x) = \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \quad \text{with} \quad |u_\nu| \leq \frac{1}{\nu}$$

is an element of the function space of bounded mean oscillation, i.e. $u \in BMO(R)$.

Suppose

$$\sum_{-\infty}^{\infty} \nu_\nu e^{i\nu x} \in BMO(R) \quad , \quad \nu_\nu \geq 0$$

then

$$\sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \in BMO(R) \quad , \quad |u_\nu| \leq \nu_\nu \cdot$$

The Hardy spaces H^p are equivalent to L_p for $p > 1$.

For $p=1$ we note $H^1 \cong BMO^*$, which can be seen as proper substitution of L_1 and L_∞ ([HAb] 4.7). Our concept above is about the alternative duality $H_{-1} \cong H_1^*$ and $H_{-1/2} \cong H_{1/2}^*$ of Hilbert spaces, embedded in a Hilbert scale framework with corresponding spectral theory instead of Banach spaces only.

Remark: In ([AZy], 5.28, 7.2, 13.11) the concept of “logarithmic”, α – capacity” of sets and convergence of Fourier series to functions with

$$\sum_1^{\infty} n[a_n^2 + b_n^2] < \infty$$

is given.

Remark: We note that in harmonic analysis the energy of the harmonic continuation $h = E(\varphi)$ to the boundary is given by

$$[\varphi]^2 := \frac{\pi}{2} \sum_1^{\infty} \nu(a_\nu^2 + b_\nu^2) = \frac{1}{2} \iint |dh(z)|^2 dx dy = \frac{1}{4\pi} \iint_{\partial B \times \partial B} \frac{|\varphi(w) - \varphi(\zeta)|^2}{|w - \zeta|^2} ds(w) d\zeta < \infty \cdot$$

From [HWe1] p. 65, §13) we recall a **Theorem of Fejér**: if the series

$$\sum_1^{\infty} n |a_n|^2$$

is convergent, then

$$f(x) := \sum_1^{\infty} a_n x^n$$

is convergent for all $x \in H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, for which for all real φ the following limit exist

$$\lim_{r \rightarrow 1} f(re^{i\varphi}) \cdot$$

In ([AZy]) the following two examples to the above are provided (see also [HEd] 9.7):

$$i) \quad \lambda(x) \approx \sum_1^{\infty} \frac{\cos 2\pi vx}{v} = -\log 2 \sin(\pi x) \quad \text{whereby} \quad \left| \sum_1^N \frac{\cos vx}{v} \right| \leq \log\left(\frac{1}{x}\right) + C ,$$

$$ii) \quad \lambda(x) \approx \sum_1^{\infty} \frac{\cos vx}{v^{1-\alpha}} \cong c_{\alpha} |x|^{-\alpha} , \quad (x \rightarrow 0; 0 < \alpha < 1) .$$

In [CBe] 8, Entry17(iv) its relationship to Ramanujan's divergent series technique is mentioned: "*Ramanujan informs us to note that*

$$\sum_1^{\infty} \sin(2\pi vx) = \frac{1}{2} \cot(\pi x) ,$$

which also is devoid of meaning" "may be formally established by differentiating the well known equality"

$$\sum_1^{\infty} \frac{\cos 2\pi vx}{v} = -\log 2 \sin(\pi x) .$$

Remark: There are several other relationships in the context of Fourier transforms and Euler's formula (see ([ETi] 2.1), [BPe]):

Let $[x]$ denote the largest integer not exceeding the real number x and let

$\rho(x) := \{x\} := x - [x]$ be the fractional part of x .

$$i) \quad \rho(x) = \{x\} = x - [x] = \frac{1}{2} - \sum_1^{\infty} \frac{\sin 2\pi vx}{2\pi|v|} ,$$

$$ii) \quad -i\pi \operatorname{sign}(x) = -2i \int_0^{\infty} \frac{\sin(tx) dt}{t} = 2 \int_0^{\infty} \frac{\sinh(tx) dt}{t} = \left[P.v. \left(\frac{1}{x} \right) \right]^{\wedge} ,$$

$$iii) \quad \sum_1^{\infty} \frac{\sin vx}{v} = \frac{\pi - x}{2} ,$$

$$iv) \quad \sum_1^{\infty} \frac{\cos vx}{v} = \frac{1}{2} \log \frac{1}{2(1 - \cos x)} , \quad 0 < x < 2\pi .$$

The Helmholtz Free Energy

In this chapter we recall the mathematical background of the Helmholtz free energy of a quantum harmonic oscillator ([RFe]). Our proposal is to move current quantum theory models from a L_2 – based to a H_{-1} – based Hilbert space environment, applying spectral theory to a corresponding self-adjoint and bounded (singular integral) operator.

The function

$$L(x) := -\log 2 \sinh(x)$$

plays a key role in the context of free energy, vacuum energy of electromagnetic fields, the density matrix for a one-dimensional harmonic oscillator and the Planck black body radiation law (concerning the notations we refer to [RFe]):

the exact value of the free energy F of a linear system of harmonic oscillators is given by

$$\beta F := \sum_{k=1}^{\infty} L(\beta \lambda_k) \quad \text{with} \quad \frac{1}{\beta} := k_B T \quad \text{and} \quad \lambda_k := \frac{\hbar \omega_k}{2}$$

with the related probability values in the form $a_k = e^{-\beta(\lambda_k - F)}$.

Due to convergence issues in order to calculate a normalization factor Z the ground state zero term $\beta \lambda_0$ is omitted and F is replaced by

$$\beta F^* = \sum_{k=1}^{\infty} \log(1 - e^{-2\beta \lambda_k}) = -\sum_{k=1}^{\infty} K(2\beta \lambda_k)$$

leading to

$$a_k^* = e^{-\beta(\lambda_k - F^*)} = \frac{1}{Z^*} e^{-\beta \lambda_k} \quad , \quad \varphi^* := \sum_{k=1}^{\infty} a_k^* \varphi_k \in H_0 \cdot$$

We propose the shift from the underlying Hilbert space H_0 into H_{-1} while keeping the information about the ground state term as part of the physical models, but applying the analysis of this paper to e.g.

$$a_k = e^{-\beta(\lambda_k - F)} = \frac{1}{Z} e^{-\beta \lambda_k} \quad , \quad \varphi := \sum_{k=1}^{\infty} a_k \varphi_k \in H_{-1}$$

and

$$Z := \|\varphi\| \cdot$$

For

$$\varphi_\lambda(x) := -\frac{1}{2\pi} \log \left[2 \sin \frac{x-\lambda}{2} \right] , \quad \lambda \in [0, x].$$

In combination with

$$\psi = \sum_{n=1}^{\infty} (\psi, \varphi_n) \varphi_n + \int \varphi_\lambda(\varphi_\lambda, \psi) d\lambda$$

and the relations (see e.g. [JNe])

$$(\varphi_n, \varphi_m) = \delta_{n,m} , \quad (\varphi_\lambda, \varphi_n) = A \varphi_n(\lambda)$$

it follows

$$\psi = \sum_{n=1}^{\infty} (\psi, \varphi_n) \varphi_n + \int \varphi_\lambda [A\psi](\lambda) d\lambda = \sum_{n=1}^{\infty} (\psi, \varphi_n) \varphi_n + A^2 \psi .$$

The spectrum for a self-adjoint operator is real and closed. If the operator is additionally compact, then the spectrum is discrete. In case the operator is not compact, but bounded (continuous), there is a spectral representation built on Riemann-Stieltjes integral over projection operator valued step functions (see also [KBr2], Lommel polynomials). In case of unbounded operators the closed graph theorem can be applied to build bounded operators with respect to the graph norm. The below indicates to analyze the graph norm for the momentum operator for those physical states, represented by the elements out of

$$H_0^\perp := \left\{ \psi \in H_{-1} := \overline{H_0} \mid \|((\psi, \varphi)) = 0, \varphi, H\varphi \in H_0 \right\}$$

whereby

$$\|\psi\|^2 := \sum_{n=1}^{\infty} |(\psi, \varphi_n)|^2 + \|A\psi\|^2 = \sum_{n=1}^{\infty} |(\psi, \varphi_n)|^2 + \|\psi\|_{-1}^2 .$$

Remark: The equivalent norm

$$\|\psi\|_*^2 = \sum_{n=1}^{\infty} |(\psi, \varphi_n)|^2 + |(\psi, \varphi_{-1/2})|^2 + \|\psi\|_{-1}^2$$

is proposed to be used to model spin effects.

Remark: We note that e.g. in case of the harmonic quantum oscillator it holds in the L_2 – framework

$$\bar{E}_0 = \frac{1}{2} \sum \hbar \omega_n \approx c \sum \hbar n = \infty \quad ,$$

which leads to the concept/requirement of “re-normalization” to ensure the existence of *bounded* Hermitian operators \bar{H}_{renorm} , with

$$\bar{H} = \bar{H}_{renorm} + \bar{E}_0 \quad .$$

This is the analogue a priori representation of a physical state of a particle in the form

$$\psi = \sum_{n=1}^{\infty} (\psi, \varphi_n) \varphi_n + \int \varphi_\lambda [A\psi](\lambda) d\lambda = \sum_{n=1}^{\infty} (\psi, \varphi_n) \varphi_n + A^2 \psi \quad .$$

The later one can be interpreted as “ideal number” or “non-standard number” as analogue to a real number r represented in the form $r + i$, whereby i denotes an infinitely small, finite non-real number, which is not equal zero, but smaller than any positive real number $\varepsilon \in R^+$ ([WLu]).

Remark: The relationship of Hermitean commutators properties with respect to the norm $\|\psi\|^2$ and the weaker $\|\psi\|_{-1}$ – norm is given by (appendix resp. [SGr] 4.384, 1.441):

- i) the norms

$$\|HA\psi\|_0^2 \cong \|A\psi\|_0^2 \cong \|\psi\|_{-1}^2$$

are equivalent

- ii) the range of a “constant” operator is zero according to

$$\frac{1}{2\pi} \int_{0 \rightarrow 2\pi} \log 2 \sin \frac{y}{2} dy = 0 \quad , \quad \frac{1}{2\pi} \int_{0 \rightarrow 2\pi} \cot \frac{\lambda - y}{2} dy = 0 \quad .$$

Appendix

From lecture notes, internet and literature

The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product (\cdot, \cdot) and the norm $\|\cdot\|$. We will consider mappings $K : H \rightarrow H$. Unless otherwise noticed the standard assumptions on K are:

- i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$
- ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,
- iii) K is injective, i.e. $Kx=0$ implies $x=0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|^2} .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$ of eigenvalues and eigenfunctions

$K\varphi_i = \lambda_i\varphi_i$ with the properties

- i) the eigenfunctions are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$
- ii) the eigenvalues tend to zero, i.e. $\lim_{i \rightarrow \infty} \lambda_i = 0$
- iii) the generalized Fourier sums $S_n := \sum_{i=1}^n (x, \varphi_i) \varphi_i \rightarrow x$ with $n \rightarrow \infty$ for all $x \in H$
- iv) the Parseval equation

$$\|x\|^2 = \sum_i (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product (\cdot, \cdot) , the norm $\|\cdot\|$ and A be a linear operator with the properties

- i) A is self-adjoint, positive definite
- ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\|^2 \quad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigenvalue of K is also an eigenvalue of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \rightarrow \lambda_i^{-1}$ we have from the previous section

- i) there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i\varphi_i, \quad (\varphi_i, \varphi_k) = \delta_{i,k} \quad \text{and} \quad \lim_{i \rightarrow \infty} \lambda_i = \infty$$

- ii) any $x \in H$ is represented by

$$(*) \quad x = \sum_{i=1}^{\infty} (x, \varphi_i) \varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{\infty} (x, \varphi_i)^2.$$

Lemma: Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i) \varphi_i, \quad \|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)^2,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i).$$

Because of (*) there is a one-to-one mapping I of H to the space \hat{H} of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, \dots)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i).$$

If we equip \hat{H} with the norm

$$\|\hat{x}\|^2 = \sum_{i=1}^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x}, \hat{y})_\alpha = \sum_i \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x}, \hat{x})_\alpha$$

Let \hat{H}_α denote the set of all sequences with finite α – norm. then \hat{H}_α is a Hilbert space. The proof is the same as the standard one for the space l_2 .

Similarly one can define the spaces H_α : they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x, y)_\alpha = \sum_i \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax, Ax)_0, \quad H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $\lim \lambda_i^\alpha \rightarrow 0$. Then there exist sequences $\hat{x} = (x_1, x_2, \dots)$ with

$$\|\hat{x}\|_2^2 < \infty, \quad \|\hat{x}\|_0^2 = \infty.$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \rightarrow H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \gamma$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \frac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

- i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$
- ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$
- iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

- i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$
- ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2(x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \quad \text{and} \quad \|x\|_2 = \|Ax\|.$$

We give three examples of differential operator and singular integral operators, whereby the integral operators are related to each other by partial integration:

Example 1: Let $H = L^2(0,1)$ and

$$Au := -u''$$

with

$$D(A) = \dot{W}_2^2(0,1) := \overset{\circ}{W}_2^1(0,1) \cap \overset{\circ}{W}_2^2(0,1).$$

Building on the orthogonal set of eigenpairs $\{\lambda_i, \varphi_i\}$ of A_i , i.e.

$$-\varphi_i'' = \lambda_i \varphi_i \quad \varphi_i(0) = \varphi_i(1) = 0$$

it holds the inclusion

$$D(A) \subseteq H_A = H_1 = \overset{\circ}{W}_2^1(0,1) \subseteq L^2(0,1).$$

Example 2: Let $H = L_{22}^*(\Gamma)$ with $\Gamma := S^1(\mathbb{R}^2)$, i.e. Γ is the boundary of the unit sphere. Then H is the space of integrable periodic function in \mathbb{R} . Let

$$(Au)(x) := -\int \log 2 \sin \frac{x-y}{2} u(y) dy =: \int k(x-y) u(y) dy$$

and

$$D(A) = H = L_{22}^*(\Gamma).$$

The Fourier coefficients of this convolution are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu$$

i.e. it holds $D(A) \subseteq H_A = H_{-1/2}(\Gamma)$.

A relation of this Fourier representation to the fractional function is given by

$$x - [x] - \frac{1}{2} = -\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$$

Remark: We give some further background and analysis of the even function

$$k(x) := -\ln \left| 2 \sin \frac{x}{2} \right| =: -\log \left| 2 \sin \frac{x}{2} \right|$$

Consider the model problem

$$\begin{aligned} -\Delta U &= 0 & \text{in } \Omega \\ U &= f & \text{on } \Gamma := \partial\Omega, \end{aligned}$$

whereby the area Ω is simply connected with sufficiently smooth boundary. Let $y = y(s) - s \in (0,1]$ be a parametrization of the boundary $\partial\Omega$. Then for fixed \bar{z} the functions

$$U(\bar{x}) = -\log|\bar{x} - \bar{z}|$$

are solutions of the Laplace equation and for any $L_1(\partial\Omega)$ -integrable function $u = u(t)$ the function

$$(Au)(\bar{x}) := \oint_{\partial\Omega} \log|\bar{x} - u(t)| dt$$

is a solution of the model problem. In an appropriate Hilbert space H this defines an integral operator, which is coercive for certain areas Ω and which fulfills the Garding inequality for general areas Ω . We give the Fourier coefficient analysis in case of $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $x(s) := (\cos(s), \sin(s))$ be a parametrization of $\Gamma := S^1(R^2)$ then it holds

$$|x(s) - x(t)|^2 = \left(\begin{array}{c} \cos(s) - \cos(t) \\ \sin(s) - \sin(t) \end{array} \right)^2 = 2 - 2\cos(s-t) = 2(1 - \cos(2\frac{s-t}{2})) = 2 \left[2\sin^2 \frac{s-t}{2} \right] = 4\sin^2 \frac{s-t}{2}$$

and therefore

$$-\log|x(s) - x(t)| = -\log 2 \left| \sin \frac{s-t}{2} \right| = k(s-t)$$

The Fourier coefficients k_ν of the kernel $k(x)$ are calculated as follows

$$k_\nu := \frac{1}{2\pi} \oint k(x) e^{-i\nu x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log \left| 2 \sin \frac{t}{2} \right| e^{-i\nu t} dt = \frac{2}{2\pi} \int_0^\pi \log \left| 2 \sin \frac{t}{2} \right| \cos(\nu t) dt = k_{-\nu}$$

As $\varepsilon \log 2 \sin \frac{\varepsilon}{2} \rightarrow 0$ partial integration leads to

$$\begin{aligned} k_\nu &= \frac{1}{\nu\pi} \sin(\nu t) \Big|_0^\pi - \frac{1}{\nu\pi} \int_0^\pi \frac{2 \sin(\nu t) \cos \frac{t}{2}}{2 \sin \frac{t}{2}} dt = -\frac{1}{\nu\pi} \int_0^\pi \frac{\sin(\frac{2\nu+1}{2}t) - \sin(\frac{2\nu-1}{2}t)}{2 \sin \frac{t}{2}} dt \\ k_\nu &= -\frac{1}{\nu\pi} \int_0^\pi \left(\frac{1}{2} + \cos(t) \dots + \cos(\nu t) \right) - \left(\frac{1}{2} + \cos(t) \dots + \cos((\nu-1)t) \right) dt = -\frac{1}{\nu} \cdot \end{aligned}$$

Extension and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1}^{\infty} e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α . Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t) \|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the (t) -norm is weaker than any α -norm. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the 0 -norm and the newly introduced (t) -norm. It holds:

Lemma: Let $\alpha > 0$ be fixed. The α -norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the α -norm, which can be reformulated in the form ($\mu, \nu > 0, \mu + \nu > 1$)

$$\|x\|_\beta^2 \leq \nu \varepsilon \|x\|_\gamma^2 + \mu e^{-\nu/\mu} \|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu .$$

The counterpart of lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

- i) $\|x - y\| \leq \|x\|$
- ii) $\|y\|_1 \leq \delta^{-1} \|x\|$
- iii) $\|x - y\|_{(t)} \leq e^{-t/\delta} \|x\| .$

Eigenfunctions and Eigendifferentials

Let H be a (infinite dimensional) Hilbert space with inner product (\cdot, \cdot) , the norm $\|\cdot\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \rightarrow L(H, H)$$

$$\lambda \rightarrow E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu(\varphi_\mu, *) d\mu \quad , \quad \mu \in [\lambda_0, \infty) ,$$

i.e.
$$dE_\lambda = \varphi_\lambda(\varphi_\lambda, *) d\lambda .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ .

The set E_λ fulfills the following properties:

- i) E_λ is a projection operator for all $\lambda \in R$
- ii) for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$
- iii) $\lim_{\lambda \rightarrow -\infty} E_\lambda = 0$ and $\lim_{\lambda \rightarrow \infty} E_\lambda = Id$
- iv) $\lim_{\substack{\mu \rightarrow \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a, b]$. Let $f : [a, b] \rightarrow R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \rightarrow H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) .$$

Symbolically one writes

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda .$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax, y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \quad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2 x, y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \quad , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x . It has the properties of a distribution function.

It holds the following eigenpair relations

$$A\varphi_i = \lambda_i\varphi_i \quad A\varphi_\lambda = \lambda\varphi_\lambda \quad \|\varphi_\lambda\|^2 = \infty, \quad (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu).$$

The φ_λ are not elements of the Hilbert space. The so-called eigendifferentials, which play a key role in quantum mechanics, are built as superposition of such eigenfunctions.

Let I be the interval covering the continuous spectrum of A . We note the following representations:

$$x = \sum_1^\infty (x, \varphi_i) \varphi_i + \int_I \varphi_\mu (\varphi_\mu, x) d\mu, \quad Ax = \sum_1^\infty \lambda_i (x, \varphi_i) \varphi_i + \int_I \lambda \varphi_\mu (\varphi_\mu, x) d\mu$$

$$\|x\|^2 = \sum_1^\infty |(x, \varphi_i)|^2 + \int_I |(\varphi_\mu, x)|^2 d\mu,$$

$$\|x\|_1^2 = \sum_1^\infty \lambda_i |(x, \varphi_i)|^2 + \int_I \lambda |(\varphi_\mu, x)|^2 d\mu$$

$$\|x\|_2^2 = \|Ax\|^2 = \sum_1^\infty \lambda_i^2 |(x, \varphi_i)|^2 + \int_I \lambda^2 |(\varphi_\mu, x)|^2 d\mu.$$

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda\varphi_\lambda(x)$$

delivers no element of the Hilbert space H , but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H . Only if one builds wave packages out of $\varphi_\lambda(x)$ it results into elements of H . The practical way to find Eigen-differentials is looking for solutions of a distribution equation.

Hermitian Operator and Physical Observables

The spectrum of a hermitian, positive definite operator

$$A : D(A) \rightarrow H$$

with domain $D(A)$ in a complex-valued Hilbert space H is discrete. This property enables an axiomatic building of the quantum mechanics, whereby, roughly speaking, physical states are modeled by the elements of the Hilbert space, observables of states by the operator A and the mean value of the observable A at the state ψ with $\|\psi\|$ is given by

$$\langle A\psi, \psi \rangle .$$

In other words, the expectation value of an operator \hat{A} is given by

$$\langle A \rangle = \int \psi^*(\vec{r}) \hat{A} \psi(\vec{r}) d\vec{r}$$

and all physical observables are represented by such expectation values. Obviously, the value of a physical observable such as energy or density must be real, so it's required $\langle A \rangle$ to be real. This means that it must be $\langle A \rangle = \langle A \rangle^*$, or

$$\langle A \rangle = \int \psi^*(\vec{r}) \hat{A} \psi(\vec{r}) d\vec{r} = \int [\hat{A} \psi(\vec{r})]^* \psi(\vec{r}) d\vec{r} = \langle A \rangle^*$$

An operator \hat{A} , which satisfy this condition are called *Hermitian*. One can also show that for a Hermitian operator,

$$\int \psi_1^*(\vec{r}) \hat{A} \psi_2(\vec{r}) d\vec{r} = \int [\hat{A} \psi_1(\vec{r})]^* \psi_2(\vec{r}) d\vec{r}$$

for any two states ψ_1 and ψ_2 .

For the eigenvalue problem of a self-adjoint, positive operator A

$$A\varphi = \lambda\varphi$$

the eigenvalues $\{\lambda\}$ are the discrete spectrum λ_n with either finite or countable infinite set of values

$$A\varphi_n = \lambda\varphi_n \quad , \quad \|\varphi_n\|^2 = 1$$

In this case the mean value of A is given by

$$\bar{A} := \langle \psi, A\psi \rangle .$$

Let w_n the probability, that the eigenvalue occurs of a measurement of the observables A then the mean value of A is defined by

$$\bar{A} := \sum_n w_n \lambda_n = \sum_n w_n \langle \varphi_n, A \varphi_n \rangle \quad \varphi = \sum_n \alpha_n \varphi_n$$

and it holds

$$\bar{A} := \langle \psi, A \psi \rangle = \left\langle \sum_n \alpha_n \varphi_n, A \left(\sum_n \alpha_n \varphi_n \right) \right\rangle = \sum_n \alpha_n^* \alpha_n \langle \varphi_n, A \varphi_n \rangle$$

$$\bar{A} := \langle \psi, A \psi \rangle = \sum_n \alpha_n^* \alpha_n \lambda_n \langle \varphi_n, \varphi_n \rangle = \sum_n |\alpha_n|^2 \lambda_n ,$$

i.e.

$$w_n = |\alpha_n|^2 = \left| \langle \varphi_n, \psi \rangle \right|^2 .$$

The general solution of the Schrödinger equation is given by

$$\varphi(\vec{x}, t) = \sum_n c_n e^{-i\lambda_n \hbar t} \varphi_n(\vec{x}) .$$

In case the operator A is only hermitian (without being positive definite) Hilbert, von Neumann and Dirac developed a corresponding spectral theory. This leads to a continuous spectrum $\lambda(\nu)$, indexed by a continuous ν . In this case $\psi(x; \nu)$ denotes an eigen function to the eigen value $\lambda(\nu)$. The norm of this function is infinite, i.e. the function is not an element of the Hilbert space. An approximation to this function with finite norm is given (with sufficiently small $\Delta \nu$) by the eigen differential

$$\Phi_{\Delta \nu}(x; \nu) = \frac{1}{\sqrt{\Delta \nu}} \int_{\nu - \Delta \nu / 2}^{\nu + \Delta \nu / 2} \phi(x; \nu') d\nu' .$$

All for the Hilbert space related properties are valid for the eigen differentials, but not for the eigenfunction itself. The scalar product of the eigenfunction is normed to a Dirac δ -function:

$$\langle \phi(x; \nu'), \phi(x; \nu'') \rangle = \delta(\nu' - \nu'') .$$

The norm of the eigen differentials is given by:

$$\langle \Phi_{\Delta \nu}(x; \nu), \Phi_{\Delta \nu}(x; \nu') \rangle = \frac{1}{\Delta \nu} \int_{\nu - \Delta \nu / 2}^{\nu + \Delta \nu / 2} \int_{\nu' - \Delta \nu / 2}^{\nu' + \Delta \nu / 2} \langle \phi(x; \eta') d\nu' \phi(x; \eta'') \rangle d\eta' d\eta''$$

$$\langle \Phi_{\Delta \nu}(x; \nu), \Phi_{\Delta \nu}(x; \nu') \rangle = \frac{1}{\Delta \nu} \int_{\nu - \Delta \nu / 2}^{\nu + \Delta \nu / 2} \int_{\nu' - \Delta \nu / 2}^{\nu' + \Delta \nu / 2} \delta(\eta' - \eta'') d\eta' d\eta''$$

The integral is 1 for $\nu = \nu'$ (with appropriate norm factor) and 0 if $|\nu - \nu'| > \Delta \nu$.

In case if ν is a momentum the eigendifferential gives a wave package with finite distance $\Delta\nu$ in the momentum space and therefore with finite distance $\Delta x \approx \frac{1}{\Delta\nu}$ in the particle space.

Such a package can normed to the value 1 (1 particle). Δx (and correspondingly $\Delta\nu$) has to be larger than all other typical distances of the problem. In this sense eigendifferentials correspond to the formalism of wave package modeling.

The eigenfunctions of the discrete and continuous spectrum build an extended Hilbert space to ensure that for every ψ it holds

$$\psi(x) = \sum_n c_n \phi_n(x) + \int c(\nu') \phi(x; \nu') d\nu'$$

with

$$c_n = \langle \phi_n(x), \psi(x) \rangle$$

and

$$c(\nu) = \langle \phi(x; \nu), \psi(x) \rangle$$

It holds the Parseval identity:

$$\langle \psi, \psi \rangle = \sum_n |c_n|^2 + \int |c(\nu')|^2 d\nu'$$

and the eigendifferential are orthogonal wave packages.

If for every function $\in L_2$ such a representation is possible, one calls the system $\{\phi\}$ a complete orthogonal system. Such a complete orthogonal system is not uniquely defined. There is always the degree of freedom

- to choose arbitrarily the phase of each eigenfunction
- the set of the non-standard eigenvalues can be orthogonalized on different ways
- to replace the index ν of the continuous spectrum by an index $\mu(\nu)$ with $\mu(\nu)$ – differentiable, monotone function of ν . Then

$$\phi(x; \mu) = \frac{\phi(x; \nu)}{\sqrt{d\mu/d\nu}}.$$

Not all existing hermitian operators are built on a complete orthogonal system of eigenfunctions. For all operators, which represent physical observables, there exists such a complete orthogonal system.

Hermite Polynomials

The weighted Hermite polynomials (e.g. [RSt] 7.6)

$$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad H_0(x) = 1, \quad H_1(x) = x,$$

form a set of orthonormal functions in $L_2(-\infty, \infty)$, i.e. the Hermite polynomials have only real zeros. As $\varphi_n \in L_2$ leading to $\psi_n = H\varphi_n \in L_2$ and

$$(\varphi_n, \psi_n) = 0, \quad L_2 := H := \text{span}[\varphi_n] = \text{span}[H(\psi_n)].$$

The Hermite polynomials $H_n(x)$ fulfill the recursion formula

$$H_n(\sqrt{2\pi}x) = 2xH_{n-1}(\sqrt{2\pi}x) - (n-1)b_n\varphi_{n-2}(x) - 2(n-1)H_{n-2}(\sqrt{2\pi}x).$$

Using the abbreviation

$$a_n := \sqrt{\frac{2(n-1)!}{n!}} \quad b_n := \sqrt{\frac{(n-2)!}{n!}}$$

this gives the recursion formula

$$\varphi_n(x) := a_n x \varphi_{n-1}(x) - (n-1)b_n \varphi_{n-2}(x), \quad \varphi_0(x) := \pi^{-1/4} e^{-\frac{x^2}{2}}, \quad \varphi_1(x) := 2^{-1/2} \pi^{-1/4} x e^{-\frac{x^2}{2}}.$$

From this the recursion formula for the corresponding Hilbert transforms of $\psi_n = H\varphi_n \in L_2$ can be calculated by

$$\hat{\varphi}_n(x) := a_n \left[x \hat{\varphi}_{n-1}(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi_{n-1}(y) dy \right] - (n-1)b_n \hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x) = \pi^{1/4} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} \sin(\omega x) d\omega.$$

An alternative polynomial system to the Hermite polynomials

We propose to apply the Lommel polynomials $g_{n,\nu+1}(x)$ as corresponding polynomial orthogonal system framework to build a (negatively scaled) Hilbert space. D. Dickinson's proof ([DDi]) of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral, enabled by the density function

$$d\psi_\nu = \frac{J_{\nu+1}(2\sqrt{x})}{\sqrt{x}J_\nu(2\sqrt{x})} dx \quad \text{with} \quad \frac{J_{\nu+1}(2\sqrt{x})}{\sqrt{x}J_\nu(2\sqrt{x})} = \lim_{n \rightarrow \infty} \frac{g_{n,\nu+1}(x)}{g_{n+1,\nu}(x)},$$

which is analytic outside any circle that contains the finite zeros of $J_\nu(1/x)$. The prize to be paid to build the orthogonality relation is an only stepwise density (bounded variation) function $d\psi_\nu$.

The Lommel polynomials $g_n(x) := g_{n,0}(x)$, defined by ([GWA] 9-6)

$$g_n(x) = \sum_0^{\lfloor n/2 \rfloor} (-1)^m \frac{(n-m)!}{m!(n-2m)!} \frac{\Gamma(n+1-m)}{m!} x^m,$$

$$h_n\left(\frac{1}{2x}\right) := x^{-n} g_n(x^2) \quad h_n\left(\frac{1}{2\sqrt{x}}\right) := x^{-n/2} g_n(x)$$

fulfill the recurrence relations

$$g_{n+1,\nu}(x) = (\nu+n+1)g_{n,\nu}(x) - xg_{n-1,\nu+1}(x), \quad g_0(x) := g_1(x) := 1.$$

$$h_{n+1,\nu}(x) = 2x(n+\nu)h_{n,\nu}(x) - h_{n-1,\nu}(x), \quad h_{-1,\nu}(x) = 0 \quad h_{0,\nu}(x) := 1$$

A relation between the modified Lommel polynomials and the Bessel function is given by Hurwitz's asymptotic formula ([GWA] 9-65):

$$J_0\left(\frac{1}{x}\right) = \lim_{n \rightarrow \infty} \frac{(2x)^{-n}}{n!} h_n(x), \quad J_0\left(\frac{2}{\sqrt{x}}\right) = \lim_{n \rightarrow \infty} \frac{x^{-n/2}}{n!} h_n\left(\frac{\sqrt{x}}{2}\right)$$

From the above and [GWA] 9-65, it follows:

$$J_0(2\sqrt{x}) = \lim_{n \rightarrow \infty} \frac{g_n(x)}{n!} = \lim_{n \rightarrow \infty} \frac{x^{n/2}}{n!} h_n\left(\frac{1}{2\sqrt{x}}\right) = \lim_{n \rightarrow \infty} \frac{x^{n/2}}{n!} \frac{1}{\sqrt{n+1}} L_n(x)$$

$$\frac{J_1(2\sqrt{x})}{\sqrt{x}} = \lim_{n \rightarrow \infty} \frac{g_{n,1}(x)}{(n+1)!}.$$

Favard's theorem ([TCh] 7, II, theor. 6.4) implies that the Lommel polynomials are orthogonal polynomials with respect to a positive weighted, bounded variation measure function. We recall from [DDi]

$$(*) \quad \sum_{k=1}^{\infty} \frac{1}{j_k^2} h_m\left(\frac{\pm 1}{j_k}\right) h_n\left(\frac{\pm 1}{j_k}\right) = \frac{\delta_{n,m}}{2(n+1)}.$$

With the relations above it follows

Proposition: For the Lommel polynomials the following orthogonality relation holds true

$$(**) \sum_{k=1}^{\infty} \frac{g_n(\alpha_k)}{2\alpha_k^{(n+1)/2}} \frac{g_m(\alpha_k)}{2\alpha_k^{(m+1)/2}} = \frac{\delta_{n,m}}{2(n+1)} .$$

Orthogonal polynomials have only real zeros and are eigenfunctions of corresponding self-adjoint differential operators. Following the arguments from §2, [DBu] and [GPO3] this property implies that the zeros of its Mellin transforms lie all on the critical line.

The proof of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral [DDi], enabled by the term

$$\frac{d\rho}{dx} := \left[J_1\left(\frac{1}{x}\right) \right] / \left[J_0\left(\frac{1}{x}\right) \right] ,$$

which is analytic outside any circle that contains the finite zeros of $J_0(1/x)$. Hence it possesses a Laurent expansion about the origin that converges uniformly on and in any annulus, whose inside boundary has the finite zeros of $J_0(1/x)$ in its interior: Let C be the contour that encircles the origin in a positive direction and that lies within the annulus.

Then it holds [DDi]

$$\frac{1}{2\pi i} \int_C x^k h_n(x) d\rho = \begin{cases} 0 & k < n \\ 1 & k = n \\ \frac{1}{2^{n+1}(n+1)} & k = n \end{cases}$$

Let $\alpha(x)$ the non-decreasing step function having increase of

$$\frac{1}{j_k^2} = \frac{1}{4\alpha_k} \quad \text{at the point} \quad x = \frac{\pm 1}{j_k} = \frac{1}{2\sqrt{\alpha_k}} \quad \text{for } k = 1, 2, 3, \dots$$

then it holds [DDi]

$$\int h_n(x) h_m(x) d\tilde{\alpha}(x) = \frac{\delta_{n,m}}{2^{n+1}(n+1)} .$$

Black-body radiation

A famous usage of Dirichlet's series is in the context of Planck's black-body radiation function

$$\frac{dR(\lambda, T)}{d\lambda} = \frac{c_1}{\lambda^5} \frac{1}{e^{c_2/\lambda T} - 1} = \frac{c_1}{\lambda^5} \sum_1^{\infty} e^{-nc_2/\lambda T}$$

with $c_1 = 2\pi^5 hc^2$ and $c_2 = hc/k$. The relation to the Zeta function

$$\zeta(s)\Gamma(s) = \int_0^{+\infty} \frac{x^s}{e^x - 1} \frac{dx}{x}$$

is given by

$$\frac{\pi^4}{90} = \zeta(4)\Gamma(4) = \int_0^{+\infty} x^4 \left(\sum_1^{\infty} e^{-nx} \right) \frac{dx}{x} = \int_0^{+\infty} x^{-4} \left(\sum_1^{\infty} e^{-\frac{n}{x}} \right) \frac{dx}{x} .$$

This describes the total radiation and its spectral density at the same time, i.e.

$$g(x)dx = \frac{x^{-4}}{e^{1/x} - 1} \frac{dx}{x} = \frac{x^4}{e^x - 1} \frac{dx}{x} = g\left(\frac{1}{x}\right)dx .$$

The weak formulation (and the positive Berry conjecture answer) should enable an alternative model for the total radiation and its spectral density.

Further information from internet

A. Einstein developed his quantum/photon concept motivated by the question: „if one moves exactly in parallel to a light signal (a photon or a wave?), how the light signal looks like? In principle it should be that the signal of light is a sequence of stationary waves, which are fixed in the time, i.e. the light signal should look like without any movement. If one follows it, it looks like a non-moving, oscillating, electromagnetic field. But something like this seems to be not existed neither caused by observation, nor by the Maxwell-equations model. The later ones exclude the existence of stationary, inelastic waves. Based on the Maxwell equations the electrons would have to lose its energy within nearly no time.

In any relativistic theory the vacuum, the state of lowest energy, if it exists in „reality“, has to have the energy zero.

In the same way for any free particle with momentum \vec{p} and mass m the energy has to be

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2} .$$

In the literature the ground state energy of the harmonic operator is mostly defined by $\frac{1}{2} \hbar \omega$.

Already M. Planck knew that this cannot be, when deriving his radiation formula: he assigned states with n photons the energy $n \hbar \omega$, but not the value

$$(n + \frac{1}{2}) \hbar \omega ,$$

which is not compatible with the relativistic co-variant description of photons.

The ground state energy is not measurable. Its chosen value is therefore arbitrarily, triggered only by the fact, to keep calculations as easily as possible, and, mainly, to ensure convergent integrals/series. Energies of freely composed systems should be additive. For photons in a box section (cavity) there are infinite numbers of frequencies ω_i . If one assigns any frequency a ground state energy value $\hbar \omega_i / 2$, then the ground state energy without photons has the infinite energy

$$\frac{1}{2} \sum_i \hbar \omega_i = \infty .$$

The **miss understanding**, that the **ground state energy is fixed** and uniquely defined, starts already in the classical physics: The definition of the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} \omega^2 x^2 =: T + V$$

defines the non-measurable ground state energy in that way, that the state of lowest energy, the point $(x=0, p=0)$ in the phase space, that the energy is zero:

the kinetic energy of strings with mass ρ are given by

$$T = \rho \int_0^l \frac{1}{2} u_x^2(x,t) dx \cdot$$

The internal forces of strings (being looked at as mechanical systems) are built on strains, depending proportionally from its lengths:

$$L = \int_0^l \sqrt{1 + u_x^2(x,t)} dx \cdot$$

For small displacements this is replaced by

$$L = l + \Delta l = \int_0^l \left[1 + \frac{1}{2} u_x^2(x,t) + \dots \right] dx \quad \text{with} \quad \Delta l \approx \int_0^l \frac{1}{2} u_x^2(x,t) dx \cdot$$

Correspondingly the potential energy $V(x)$ is approximately defined by

$$V(L) = V(l + \Delta l) \approx V(l) + \Delta l \left. \frac{dV}{dL} \right|_{L=l} \cdot$$

Putting

$$\sigma_s := \left. \frac{dV}{dL} \right|_{L=l}$$

as “tension” or “strain constant”, the choice

$$V(l) := 0$$

simplifies the algebraic term for the potential energy V in the form:

$$V \approx \sigma_s \int_0^l \frac{1}{2} u_x^2(x,t) dx \cdot$$

For example for the “string velocity”

$$c_s := \sqrt{\frac{\sigma_s}{\rho}}$$

the wave equation of strings is given by

$$u_{tt} - c_s^2 u_{xx} = 0 \cdot$$

Alternatively to $V(x)$ in case of the harmonic oscillator one could have chosen instead e.g.

$$V(x) = \frac{1}{2} \omega^2 x^2 - \hbar \omega / 2$$

or (with reference to the theory of minimal surfaces, using $1 + \sinh^2 x = \cosh^2 x$)

$$1 + V(x) = \kappa \cosh x .$$

For a single particle in a potential energy $V(x,t)$ the Schrödinger equation is ([RFe] 4-1)

$$\psi_t(x,t) = -\frac{i}{\hbar} \bar{H} \psi(x,t)$$

with

$$\bar{H} \psi(x,t) := -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + V(x,t) .$$

With respect to our proposal above we note

$$\bar{H}x - x\bar{H} = -\frac{\hbar^2}{m} \frac{\partial}{\partial x} \quad \text{resp.} \quad \|(\bar{H}x - x\bar{H})\psi\|_{-1} = c\|\psi\|_0 .$$

References

- [HAb] H. Abels, Pseudo-Differential and Singular Integral Operators, De Gruyter, Berlin/Boston, 2011
- [JAh] J. F. Ahner, Some spectral properties of an integral operator in potential theory, Proc. Edinburgh Math. Soc. (1986) 29, 405-411
- [CBe] B. C. Berndt, Ramanujan's Notebooks, Part I, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985
- [MBe] M. V. Berry, J. P. Keating, $H = xp$ and the Riemann zeros, in Supersymmetry and Trace Formulae: Chaos and Disorder (Ed. I.V. Lerner, J.P. Keating, D.E.Khmel'nitski), Kluver, New York, 355-367 (1999)
- [KBr0] K. Braun, Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations, Jap. Journal of Applied Mathematics, 3, 1, 59-72, (1986)
- [KBr1] K. Braun, A spectral analysis argument to prove the Riemann Hypothesis, www.riemann-hypothesis.de
- [KBr2] K. Braun, A Note to the Bagchi Formulation of the Nyman RH criterion, www.riemann-hypothesis.de
- [KBr3] K. Braun, Some remarkable Pseudo-Differential Operators with order -1, 0, 1, <http://www.navier-stokes-equations.com/>
- [WBU] W. Bulla, F. Gesztesy, Deficiency indices and singular boundary conditions in quantum mechanics, J. Math. Phys. 26, 2520-2528 (1985)
- [HCA] H. Cartan, Differential Forms, Dover Publications, Inc., Mineola, New York, 2006
- [PCo] P. Constantin, On the Euler equations of incompressible fluids, Bull. Amer. Math. Soc. 44 (2007), 603-621
- [TCh] T.S. Chihara, An introduction to Orthogonal Polynomials, Mathematics and its Applications 13, Gordon and Breach, New York, 1978
- [KDe] K. Deimling, Nonlinear Functional Analysis, Dover Publications, Inc., Mineola, New York, 2009
- [DDi] D. Dickinson, On Lommel and Bessel polynomials, Proc. Amer. Math. Soc. 5, pp. 946-956 (1954)
- [SDo] S. K. Donaldson, Lectures on Lie groups and geometry, March 25, 2011
- [RDu] R. J. Duffin, Hilbert Transforms in Yukawan Potential Theory, Proc. Nat. Acad. Sci. USA, 69, No 12, pp. 3677-3679 (1972)
- [HEd] H.M. Edwards, Riemann's Zeta Function, Dover Publications, Inc., Mineola, New York, 1974
- [GEs] G. I. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, American Math. society, Providence, Rhode Island (Russian original edition: 1973)
- [RFe] R.P. Feynman, A.R. Hibbs, Quantum Mechanics and Path Integrals, Dover Publications, Inc., Mineola, New York, 1965

- [HFI] H. Flanders, *Differential Forms with Applications to the Physical Science*, Dover Publications, Inc., New York, 1989
- [DGa] D. Gaier, *Konstruktive Methoden der konformalen Abbildung*, Springer-Verlag, Berlin, 1964
- [LGa] L. Garding, *Some points of analysis and their history*, University Lecture Series, no. 11, American Mathematical society, Providence, R.I., 1991
- [SGr] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals Series and Products*, Fourth Edition, Academic Press, New York, San Francisco, London, 1965
- [TGr] T. H. Gronwall, Über die Gibbssche Erscheinung und die trigonometrischen Summen $\sin x + 4 \dots \sin 2x + \dots + 1/n \sin nx$, *Mathematische Annalen*, ISSN 1432-1807, Vol. 72 (2. 1912), p. 228-243
- [WHe] W. Heisenberg, "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen". *Zeitschrift für Physik* 33 (1) (1925) 879–893
- [JHe] J. G. Heywood and O. D. Walsh, A counterexample concerning the pressure in the Navier-Stokes equations, as $t \rightarrow 0^+$, *Pacific J. Math.* Volume 164, Number 2 (1994), 351-359
- [SKI] S. Klainerman, I. Rodnianski, Regularity and geometric properties of solutions of the Einstein-Vacuum equations, *J. Equ. aux der. part.*, Forges-les-E., 3, 7 juin 2002
- [SLi] S. Lie, *Über die Grundlagen der Geometrie*, Wissenschaftliche Buchgesellschaft, Darmstadt, Sonderausgabe MCMLXVII
- [ILi] I. K. Lifanov, L. N. Poltavskii, G. M. Vainikko, *Hypersingular Integral Equations and their Applications*, Chapman & Hall/CRC, Boca Rato, London, New York, Washington, D. C., 2004
- [Ili1] I. K. Lifanov, A. S. Nenashev, Generalized functions on Hilbert spaces, singular integral equations, and problems of aerodynamics and electrodynamics, *Differential Equations*, 2007, Band: 43, Heft 6, 862-872
- [HLo] H. Looman, Über die Cauchy-Riemannschen Differentialgleichungen, *Göttinger Nach*, (1923) 97–108
- [SMo] S. Montgomery-Smith, M. Pokorný, A counter example to the smoothness of the solution to an equation arising in fluid mechanics, *Comment. Math. Univ. Carolin.*, 43, I (2002) 61-75
- [JNe] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin, Springer, 1932
- [JNe] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin, Springer, 1932
- [NNi] N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Leipzig, B.G. Teubner Verlag, 1906
- [JNi] J.A. Nitsche, Finite Element Methods for Conformal Mapping, *SIAM J. Num. Anal.*, 26, 6, (1989) 1525-1533

- [JNi1] J.A. Nitsche, Direct Proofs of Some Unusual Shift-Theorems, Anal. Math. Appl., Gauthier-Villars, Montrouge, (1988) 383-400
- [POI] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer, 1991
- [BPe] B. E. Petersen, Introduction to the Fourier Transform & Pseudo-Differential Operators, Pitman Publishing Limited, Boston, London, Melbourne
- [JPI] J. Plemelj, Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911
- [GPo] G. Polya: Collected Papers, Volume II: location of zeros, Edited by R. P. Boas, The MIT Press, 1974
- [LPr] L. Prandtl, A. Betz, Vier Abhandlungen zur Aerodynamik und Hydrodynamik., <http://webdoc.sub.gwdg.de/univerlag/2010/GKSM3.pdf>
- [HRo1] H. Rollnik, Quantentheorie 1, Springer Verlag, Berlin, Heidelberg, New York, 1995
- [HRo2] H. Rollnik, Quantentheorie 2, Springer Verlag, Berlin, Heidelberg, New York, 2002
- [BRi] B. Riemann, Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe, Habilitationsschrift
- [CRu] C. Runge, Über eine Analogie der Cauchy-Riemannschen Differentialgleichungen in drei Dimensionen, Gött. Nachr. (1922) 129-136
- [MRu] M. Ruzicka, Nichtlineare Funktionalanalysis, Springer-Verlag, Berlin, Heidelberg, New York, 2003
- [HSc] H. Schlichting, Grenzschicht-Theorie, Verlag G. Braun, Karlsruhe, 1982
- [ESc] E. W. Schipolski, Atomphysik, Teil I, VEB Deutscher Verlag der Wissenschaften, Berlin 1970, original Russian edition 1944
- [ESc] E. Schrödinger, About Heisenberg Uncertainty Relation, Proceedings of The Prussian Academy of Sciences, Physics-Mathematical Section. 1930. XIX, pp.296-303, <http://www.cmi.ac.in/~debangshu/schrodinger.pdf>
- [ESc1] E. Schrödinger, Space-Time Structure, Cambridge University Press, 1950
- [RSe] R. U. Sexl, H. K. Urbantke, Gravitation und Kosmologie, Spektrum Akademischer Verlag, Heidelberg, Berlin, Oxford, 1995
- [DSh] D. Shirokoff, A Pressure Poisson Method for incompressible Navier-Stokes Equations, internet
- [ESt] E. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, New Jersey, 1970
- [ESt1] E. M. Stein, Harmonic Analysis, Princeton University Press, Princeton, New Jersey, 1993
- [RSt] R.S. Strichartz, A Guide to Distribution Theory and Fourier Transforms, World Scientific, New Jersey,, London, Singapore, Hongkong, reprinted 2008

- [DSt] D. J. Struik, Lectures on Classical Differential Geometry, Dover Publications Inc., New York, 1988
- [ETi] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford University Press Inc., New York, first published 1951, Second Edition 1986
- [FTr] F. G. Tricomi, Integral Equations, Dover Publications Inc., New York, 1985
- [WVe] W. Velte, Direkte Methoden der Variationsrechnung, B. G. Teubner, Stuttgart, 1976
- [GWa] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 2nd edition first published 1944, reprinted 1996, 2003, 2004, 2006
- [HWe] H. Weyl, "Das gruppentheoretische Fundament der Tensorrechnung." Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1924
- [HWe1] H. Weyl, E. Landau, B. Riemann, Das Kontinuum und andere Monographien, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 2006
- [HWe2] H. Weyl, Space, Time, Matter, Dover Publications Inc., 1921
- [AZy] A. Zygmund, Trigonometric series, Vol. I, Cambridge University Press, 1968